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Titre de la thèse :

Sur une méthode numérique ondelettes/domaines ficitfs lisses pour l'approximation de problèmes de Stefan

Soutenue le , devant le jury :

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Résumé: Notre travail est consacré à la définition, l'analyse et l'implémentation de nouveaux algorithmes numériques pour l'approximation de la solution de problèmes à 2 dimensions du type problème de Stefan. Dans ce type de problèmes une équation aux dérivée partielle parabolique posée sur un ouvert ω quelconque est couplée avec une autre équation qui contrôle la frontière γ du domaine lui même.

Les difficultés classiquement associés à ce type de problèmes sont: la formulation en particulier celle de l'équation pour le bord du domaine; l'approximation de la solution liées à la forme quelconque du domaine; les difficultés associées à l'implication des opérateurs de trace(approximation, conditionnement).

De plus, de nombreuses situations d'intérêt physique par exemple demandent des approximations de haut degré.

Notre travail s'appuie sur une formulation de type espaces de niveaux(level set) pour l'équation du domaine, et une formulation de type domaine fictif(Ω) pour l'équation initiale. Le contrôle des conditions aux limites est effectué à partir de multiplicateurs de Lagrange agisant sur une frontière(Γ) dite de contrôle différente de la frontière(γ) du domaine(ω). L'approximation est faite à partir d'un schema aux différences finies pour les dérivées temporelle et une discrétisation à l'aide d'ondelettes bi-dimensionelles pour l'équation initiale et unidimensionnelle pour les multiplicateurs de Lagrange. Des operateurs de prolongement de ω à Ω sont également construits à partir d'analyse multiéchelle sur l'intervalle.

Nous obtenons ainsi: une formulation pour laquelle l'existence de la solution est démontrée; un algorithme convergent pour lequelle une estimation globale d'erreur(sur Ω) est établie; une estimation intérieure pourtant sur l'erreur dans un domaine ω_0, $\overline{\omega_0} \subset \omega$; des estimations sur les conditionnement associés à l'opérateur de trace; des algorithmes de prolongement régulier.

Différentes expériences numériques en 1D ou 2D sont effectuées en particulier un problème de Stefan à 2 phases.

Mots clés: méthode de domaine fictif, problème de Stefan, prolongement, multiplicateurs de Lagrange, méthode de Petrov-Galerkin, approximation par ondelettes,

préconditionnement.

Abstract: Our work is devoted to the definition, analysis and implementation of a new algorithms for numerical approximation of the solution of 2 dimensional Stefan problem. In this type of problem a parabolic partial differential equation defined on an open set Ω is coupled with another equation which controls the boundary γ of the domain itself.

The difficulties traditionally associated with this type of problems are: the particular formulation of equation on the boundary of domain; the approximation of the solution defined on general domain; the difficulties associated with the involvement of trace operation (approximation, conditioning).

In addition, many situations of physical interest, for example, require approximations of high degree.

Our work is based on a formulation of type level set for the equation on the domain, and a formulation of type fictitious domain (Ω) for the initial equation. The control of boundary conditions is carried out through Lagrange multipliers on boundary (Γ), called control boundary, which is different with boundary (γ) of the domain (ω). The approximation is done by a finite difference scheme for time derivative and the discretization by bi-dimensional wavelet for the initial equation and one-dimensional wavelet for the Lagrange multipliers. The extension operators from ω to Ω are also constructed from multiresolution analysis on the interval.

We also obtain: a formulation for which the existence of solution is demonstrated; a convergent algorithm for which a global estimate error(on Ω) is established; interior error estimate on domain ω_0, $\overline{\omega_0} \subset \omega$; estimates on the conditioning related to the trace operator; algorithms of smooth extension.

Different numerical experiments in 1D or 2D are implemented, in particular the 2 phases Stefan problem.

Key words: fictitious domain method, Stefan problem, extension, Lagrange multipliers, Petrov-Galerkin method, wavelet approximation, preconditioning

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Contents

Notations

General notations

Functions

$\phi, \tilde{\phi}$: primal and dual scaling function.

$\psi, \tilde{\psi}$: primal and dual wavelet.

$\forall f \in L^2(I\!R),\ f_{j,k}(x) = 2^{j/2} f(2^j x - k)$.

$\phi_{j,k}(\phi_k^j), \tilde{\phi}_{j,k}$: primal and dual scaling function at resolution 2^{-j}.

$\psi_{j,k}(\psi_k^j), \tilde{\psi}_{j,k}$: primal and dual wavelet at resolution 2^{-j}.

Φ_α: multivariate scaling function(for bivariate situation $\alpha = (j, k, k') \in Z\!\!Z^3$).

Ψ_α^λ: multivariate wavelet.

c_k^j: scaling coefficient $c_k^j =< f, \phi_{j,k}) >_{L^2}$.

d_k^j: wavelet coefficient $d_k^j =< f, \psi_{j,k}) >_{L^2}$.

$\{h_n\}, \{\tilde{h}_n\}$: filter associated to primal and dual scaling function.

$\{g_n\}, \{\tilde{g}_n\}$: filter associated to primal and dual wavelet.

$m_0, \tilde{m}_0$: symbol of primal and dual scaling function.

N_m: B-spline function of order m.

Spaces

$C^m (m \in I\!N)$: space of m-times continuously differentiable functions.

L^P: Lebesgue space equipped with the norm $\|f\|_{L^p} = (\int |f|^p)^{1/p}$.

H^s: Sobolev space of order s equipped with the norm $\|f\|_{H^s} = (\sum_{|\alpha| \leq s} \|f^\alpha\|_{L^2}^2)^{1/2}$.

V_j: multiresolution space at scale j.

W_j: detail space at scale j.

$< f, g >_{L^2(\Omega)} = \int_\Omega f\bar{g}$, scalar product on $L^2(\Omega)$.

Fourier transform

We note $\hat{f}$ the Fourier transform of f defined as follows:

(1) $\forall f \in L^2(I\!R)$, $\hat{f}(\xi) = \int_{-\infty}^{\infty} dx e^{-i2\pi x\xi} f(x)$,

(2) $\forall f \in L^2([0,1])$, $\hat{f}(\xi) = \int_0^1 dx e^{-i2\pi x\xi} f(x)$,

(3) $\forall f \in l^2(Z\!\!Z/NZ\!\!Z)$, $\hat{f}(k) = \frac{1}{N}\sum_{j=0}^{N-1} f_j e^{-i2\pi kj/N}$, $k = 0, 1, \cdots, N-1$.

Fictitious domain method

Spaces

ω: original domain with boundary γ.

Ξ: control domain with boundary Γ.

Ω: fictitious domain, $\Omega \subset I\!R$ or $\Omega \subset I\!R^2$.

Typical situation in $I\!R^2$:

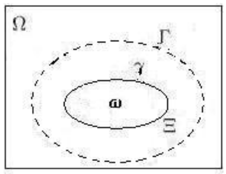


Figure 1: Geometry of fictitious domain method

$X(\omega)$, $X(\Omega)$, $Y(\gamma)$: Hilbert spaces on ω, Ω, γ.

$Y'(\gamma)$: dual space of $Y(\gamma)$.

$X_g(\omega)$: $\{v \in X(\omega) \mid T_\omega(v) = g\}$, when $g = 0$, it is noted $X_0(\omega)$.

X_h, $\tilde{X}_h$: finite dimensional subspace of $X(\Omega)$(usually $h = 2^{-j}$).

Y'_h: finite dimensional subspace of $Y(\gamma)$(usually $h = 2^{-j'}$).

Operators

T_ω: trace operator mapping $X(\omega)$ onto $Y(\gamma)$.

T_γ: trace operator mapping $X(\Omega)$ onto $Y(\gamma)$.

a_ω, a_Ω: bilinear forms on $X(\omega)$, $X(\Omega)$.

l_ω, l_Ω: linear forms on $X(\omega)$, $X(\Omega)$.

R_ω, $R_{\Omega\setminus\bar{\omega}}$: restriction operators from $X(\Omega)$ to ω and $\Omega \setminus \bar{\omega}$.

Analysis of a smooth fictitious domain method

Spaces

U_h^Ω, V_h^Ω: finite dimension subspaces of $H^1_{\mathcal{P}}(\Omega)$ which is the periodic subspace of $H^1(\Omega)$.

$Q_{h'}^\Gamma$, $Q_{h''}^\gamma$: finite dimension subspaces of $H^{-\frac{1}{2}}(\Gamma)$, $H^{-\frac{1}{2}}(\gamma)$ respectively.

Operators

L: second-order partial differential operator.

L^*: adjoint of the operator L.

$\triangle$: Laplacien operator $\triangle = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$.

$a_\Omega(\cdot,\cdot)$: bilinear form associated to L, $a(u,v) = (Lu, v)$, $u, v \in H^1(\Omega)$.

R_Ξ, R_ω: restriction operators from $H^1(\Omega)$, respectively, to $H^1(\Xi)$ and $H^1(\omega)$.

T_Γ, T_γ: trace operators from $H^1(\Omega)$, $H^1(\Omega)$ respectively, to $H^{\frac{1}{2}}(\Gamma)$, $H^{\frac{1}{2}}(\gamma)$.

$P_{V_h^\Omega}$: biorthogonal projector from $L^2(\Omega)$ on V_h^Ω.

$P_{U_h^\Omega}$: orthogonal projector from $L^2(\Omega)$ on U_h^Ω.

$K_j = \{\alpha = (j, k_1, \cdots, k_n), k_i = 0, \cdots, 2^j - 1, i = 1, \cdots, n\}$ with $\mid \alpha \mid = j$.

$K'_{j'} = \{\alpha' = (j', l'_1), l'_1 = 0, \cdots, 2^{j'} - 1\}$ with $\mid \alpha' \mid = j'$.

$K''_{j''} = \{\alpha'' = (j'', s''_1), s''_1 = 0, \cdots, 2^{j''} - 1\}$ with $\mid \alpha'' \mid = j''$.

Functions

$\tilde{f}$: an extension function from $L^2(\omega)$ to $L^2(\Omega)$.

ϕ_α^Ω: orthogonal scaling functions of a multi-resolution analysis of $H^1_{\mathcal{P}}(\Omega)$, $\alpha \in K_j$.

$\theta_\alpha^\Omega = (L^*)^{-1}\phi_\alpha^\Omega$, $\alpha \in K_j$.

$\tilde{\theta}_\alpha^\Omega = L\phi_\alpha^\Omega$, $\alpha \in K_j$.

$\phi_{\alpha'}^\Gamma$: orthogonal scaling functions of a multi-resolution analysis of $H^{-\frac{1}{2}}(\Gamma)$, $\alpha' \in K'_{j'}$.

$\phi_{\alpha''}^\gamma$: orthogonal scaling functions of a multi-resolution analysis of $H^{-\frac{1}{2}}(\gamma)$, $\alpha'' \in K''_{j''}$.

(u, λ): exact solution of original problem, Lagrange multiplier.

$(u_h, \lambda_{h'})$: approximation of (u, λ).

$(U_h, \Lambda_{h'})$: vector of coordinates of $(u_h, \lambda_{h'})$ on the bases $\{\phi^{\Omega}_{\alpha}, \alpha \in K_j\}$, $\{\phi^{\Gamma}_{\alpha'}, \alpha' \in K'_{j'}\}$.

Multivariate situation:

C: $C_{\alpha\alpha'} = (\zeta\theta^{\Omega}_{\alpha}, \phi^{\Gamma}_{\alpha'})_{L^2(\Gamma)}$, $\alpha \in K_j$, $\alpha' \in K'_{j'}$

D: $D_{\alpha''\alpha} = (B\phi^{\Omega}_{\alpha}, \phi^{\gamma}_{\alpha''})_{L^2(\gamma)}$, $\alpha \in K_j$, $\alpha'' \in K''_{j''}$

F_h: $(F_h)_{\alpha} = (\tilde{f}, \theta^{\Omega}_{\alpha})_{L^2(\Omega)}$, $\alpha \in K_j$

G_h: $(G_h)_{\alpha''} = (g, \phi^{\gamma}_{\alpha''})_{L^2(\gamma)}$, $\alpha'' \in K''_{j''}$

where $\zeta = T_{\Gamma} R_{\Xi}$, $B = T_{\gamma} R_{\omega}$.

$\mathrm{cond}_2(A)$: condition number of matrix A related to the norm $\| \cdot \|_2$

Smooth extension

Spaces

$V_j[a, b]$: Multiresolution space of $L^2([a, b])$.

Functions

$\phi_{j,k}(x)$: Interior scaling functions at scale j, $k = N, \cdots, 2^j - N - 1$.

$\phi^0_{j,k}(x)(\phi^1_{j,k}(x))$: Left(right) edge scaling functions, $k = 0, \cdots, N - 1$.

$\psi_{j,k}(x)$: Interior wavelets at scale j, $k = N, \cdots, 2^j - N - 1$.

$\psi^0_{j,k}(x)(\psi^1_{j,k}(x))$: Left(right) edge wavelets, $k = 0, \cdots, N - 1$.

$\tilde{\phi}^0_{j,k}(x)$: The function on which starts the construction of the left edge scaling function on arbitrary interval $[a, b]$, reads $2^{\frac{j}{2}} \sum_{n=k}^{2N-2} \binom{n}{k} \phi(2^j B^{-1}x + n - N + 1)\chi_{[a,b]}$ with B a map from $[0, 1]$ onto $[a, b]$.

$\hat{\phi}^0_{j,k}(x) \in C^{0.2N}$ smooth extension of $\tilde{\phi}^0_{j,k}(x)$, with $\hat{\phi}^0_{j,k}(x) = 2^{\frac{j}{2}} \sum_{n=k}^{2N-2} \binom{n}{k} \phi(2^j B^{-1}x + n - N + 1)$.

$\tilde{\phi}^1_{j,k}(x)$: The function on which starts the construction of the right edge scaling function on arbitrary interval $[a, b]$, reads $2^{\frac{j}{2}} \sum_{n=k}^{2N-2} \binom{n}{k} \phi^{\sharp}(2^j(1 - B^{-1}x) + n - N + 1)\chi_{[a,b]}$ with $\phi^{\sharp}(x) = \phi(1 - x)$.

$\hat{\phi}^1_{j,k}(x) \in C^{0.2N}$ smooth extension of $\tilde{\phi}^1_{j,k}(x)$, with $\hat{\phi}^1_{j,k}(x) = 2^{\frac{j}{2}} \sum_{n=k}^{2N-2} \binom{n}{k} \phi^\sharp(2^j(1-B^{-1}x) + n - N + 1)$.

Stefan problem

Spaces

Ω: Initial domain separated into two subdomains ω and ω^c associated to the different phases.

γ_n: Interface between ω and ω^c at time $n\delta t$.

$\Gamma^{\mathrm{out}}_n(\Gamma^{\mathrm{in}}_n)$: Outer control boundary(interior control boundary) at time $n\delta t$.

Functions

η: Level set function.

Introduction générale

Ce travail est consacré à la définition, l'analyse et l'implémentation d'un nouvel algorithme numérique pour l'approximation des équations elliptiques sur des domaines généraux et à son application à les problèmes de Stefan. Ces problèmes sont utilisés pour modéliser des phénoménes physiques polyphasiques, tels que la fonte de la glace, la croissance cristalline ou la solidification dendritique [11],[1],[44]. Ils apparaissent aussi dans les problèmes d'optimisation de forme[42]. D'un point de vue mathématique, les problèmes de Stefan renvoient à un système d'équations aux dérivées partielles régissant simultanément des quantités sur un ouvert ω/ou un famille d'ouverts et la frontière γ de cet ouvert/ou leur frontières . Des résultats généraux d'existence et d'unicité de solutions peuvent être trouvés par exemple dans [31] et [29]. Habituellement, il est supposé que les équations aux dérivée partielles posées sur l'ouvert ω sont paraboliques ou elliptiques tandis que celles qui gouvernent la frontière sont hyperboliques.

Plusieurs méthodes numériques spécifiques ont été développées pour approcher la solution des problèmes de Stefan. Une première difficulté vient de la représentation de la frontière γ, appelée également interface. Les méthods de Front tracking[44] utilisent une représentation explicite de l'interface et se heurtent à des changements topologiques liés à l'agregation ou au morcellement des différents phases. Les méthodes implicites telles que les méthodes d'ensemble de niveau [14] ou les méthodes de champ de phase[46] sont généralement plus robustes. Dans un modèle de champ de phase, le domaine est paramétré par une fonction régulière constante dans chaque domaine associé à une phase, et varie rapidement dans une région dite d'interface. La méthode de surface de niveau introduit une fonction de niveau telle que l'interface est représentée par l'ensemble d'isovaleur nulle. Elle a été appliquée à de nombreux problèmes. En général, l'évolution de l'interface est régie par les gradients normaux des variable à l'interface. Une estimation précise de ces gradients est donc un point crucial pour la qualité d'une simulation numérique.

Une deuxième difficulté est liée à la forme du domaine ω et l'approximation des fonctions sur ces domaines. Des méthodes de domaine fictif[3] sont souvent utilisées et consistent à prolonger le domaine initial dans un domaine Ω plus grand mais plus simple. Différentes approches parmi lesquelles des méthodes de pénalisation[2],[45] avec multiplicateurs de Lagrange[41] ont été développées. En général, une extension de l'opérateur aux dérivées partielles doit alors être construit et diverses difficultés apparaissent liées à la régularité des extensions et à l'efficacité de ces techniques. Les conditions aux limites à l'interface impliquent en particulier les opérateurs de trace qui sont connus pour être mal conditionnés.

La méthode que nous allons présenter et analyser a été développée pour faire face aux différents inconvénients des méthodes existantes. Elle est construite sur une approche de type level set pour l'équation de la frontière et une formulation de type domaine fictif impliquant des multiplicateurs de Lagrange pour l'équations aux dérivées partielles initiale. L'approximation est principalement basée sur une méthode de Petrov-Galerkin utilisant des ondelettes biorthogonales sur le domaine fictif et des ondelettes orthogonales pour la représentation des multiplicateurs de Lagrange. Dans la continuité des travaux de J. Haslinger et al [40], nous introduisons une nouvelle frontière Γ qui joue le rôle d'une frontière de contrôle ou les multiplicateurs de Lagrange sont appliqués. Cette formulation rapproche nos travaux de probème de contrôlabilité d'équations elliptiques par une frontière. La géométrie globale est visible sur la figure 2. On rappelle que:

–ω est le domaine initial sur lequel est posé le problème initial, sa frontière γ étant elle même une partie de la solution.

–Γ est la frontière de contrôle

–Ω est le domaine fictif

–Ξ est le domaine de contrôle incluant ω

–ω_0 est un domaine intérieur à $\omega(\bar{\omega}_0 \subset \omega)$

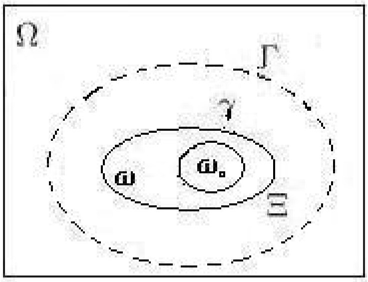


Figure 2: Geometry

Comme il sera montré dans ce travail, l'avantage de l'introduction de cette nouvelle frontière est que la perte de régularité de la solution du problème s'effectue alors sur Γ. Cette perte de régularité affecte pas la qualité de l'approximation à l'intérieur du domaine d'origine ω. Nous nous concentrerons également sur le problème de l'extension et mettrons l'accent sur l'importance de la régularité d'extension sur γ. La méthode complète que nous proposons est appelée méthode ondelettes/domaine fictif lisse. Nous allons d'abord l'analyser pour une équation elliptique puisque, comme il sera expliqué plus tard, la discrétisation en temps utilisé pour l'approximation de l'équation parabolique d'un problème de Stefan transforme le problème en une cascade de problèmes elliptiques sur un domaine fixe mais arbitraire.

Nous allons analyser la méthode de domaine fictif lisse pour le problème elliptique en nous concentrant sur les aspects suivants:

—-existence et approximation de la solution: Nous appliquons le résultat d'existence donné par J. Haslinger et al[40] dans la situation continue et discrete. Notons que ce résultat contraint la régularité de la condition aux limites sur la frontière γ à un sous-espace stricte(mais dense) de $H^{1/2}(\gamma)$. de la formulation faible. Pour l'approximation, nous utilisons une méthode de Petrov-Galerkin méthode qui transforme la matrice de rigidité en la matrice identité. L'avantage d'une telle méthode est clairement son implémentation numérique. On donne la condition de Ladyšenskaja-

Babuška-Brezzi (LLB) qui joue un rôle important dans l'estimation d'erreur et conditionnement. Elle décrit la compatibilité entre l'espace de discrétisation sur Ω, l'espace de discrétisation sur Γ et l'espace de discrétisation sur γ.

—-estimations d'erreur globale: Cela comprend l'analyse des taux de convergence $u - u_h$ et de $\lambda - \lambda_h$(l'indice h représentant l'approximation, u étant la solution et λ les multiplicateurs de Lagrange). Ils dépendent de la régularité globale de u sur Ω. Le manque de régularité à la limite de contrôle fait que la décroissance des erreurs globales avec h, est limitée.

—-estimation de l'erreur intérieur sur un sous-domaine $\omega_0 \subset \omega$: Les résultats présentés ici sont issus d'une approche spécifique développée dans [56] , [9]. Le fait que nous utilisons une Petrov-Galerkin méthode d'ondelettes et le fait que les bases à support compact ne peuvent pas être utilisés dans tous les espaces rend la dérivation des résultats plus difficile.

—-estimation du conditionnement de la matrice itérative: En dimension 1, nous utilisons un théoréme de plongement pour prouver que le conditionnement du problème est indépendant du niveau de discrétisation j sur $\Omega(h = 2^{-j})$. En dimension 2, ce donditionnement est contrôlé par $c2^{2j+j'+j''}$, ou c est une constante et j, j', j'' désigne respectivement le niveau de discretizaion sur Ω, Γ, γ. Cette estimation améliore celle obtenue dans [6] ou $j' = j''$ pour la méthode classique de domaine fictif utilisant $\Gamma = \gamma$.

—-prolongement lisse. Nous proposons des procédures de prolongement utilisant les ondelettes à support compact dans deux situations: Soit la fonction à prolonger est connue sur tous les points de ω, soit cette fonction n'est connue que sur une grille fixe à l'intérieur du domaine ω. Pour la premiere situation, la procédure est basée sur la construction des ondelettes à support compact sur le intervalle $[0, 1]$ [21] avec restriction sur l'échelle de j ($2^j \geq 2N$, N étant le nombre de moments nuls). Pour la seconde situation, nous avons besoin de faire l'extension à partir $p + 1(p \geq 0)$ points connus sur $[0, 1]$. Comme $p \geq 2N$ n'est pas assurée, nous utilisons le travail des S. Bertoluzza et al [10] concernant les ondelettes sur $[0, 1]$

au grandes échelles. La procédure est généralisée à un intervalle arbitraire $[a, b]$ et au cas 2D. La technique de prolongement lisse convient naturellement quand on applique la méthode de domaine fictif lisse.

Tous ces algorithmes sont fusionnés en un programme complet pour la simulation d'un problème de Stefan à deux phases. La discrétisation en temps et l'approximation de l'équation de level set sont présentés. Notons que notre méthode appliquée à l'approximation d'un problème de Stefan à deux phase implique, à chaque temps $n\delta t$, la définition de deux frontières de contrôle, Γ_n^{in} and Γ_n^{ex} afin de contrôler la frontière mobile γ_n à la fois par l'intérieur et par l'extérieur(voir la figure 3).

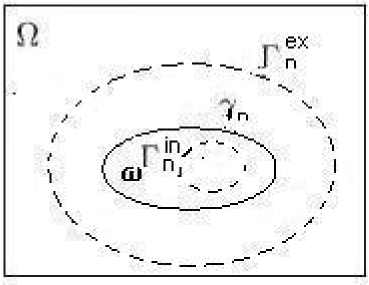


Figure 3: Γ_n^{in}: frontière de contrôle intérieur à temp $n\delta t$, Γ_n^{ex}: frontière de contrôle extérieure à temp $n\delta t$, γ_n: frontière mobile à temp $n\delta t$

Le manuscrit est organisé comme suit:

Le premier chapitre rappelle la construction et les propriétés importantes des l'analyse multirésolution et des ondelettes, ainsi que les algorithmes numériques associés à la leur utilisation pour l'approximation de l'opérateur aux dérivées partielles.

Le deuxième chapitre donne un apercu des méthodes classiques de domaine fictif approchée par des méthodes de Galerkin ou de Petrov-Galerkin. Nous détaillons en particulier les limitations de ces méthodes et en déduisons les différents directions de notre travail.

Le troisième chapitre présente notre nouvelle méthode de domaine fictif, appelée méthode ondelettes/domaine fictif lisse. L'approximation est réalisé par une méthode d'ondelettes de type Petrov-Galerkin et la condition aux limite est appliquée en utilisant une frontière de contrôle. Cette section contient l'analyse théorique et l'implémentation numérique. Différents avantages de cette méthode sont mis en envidence théoriquement et numériquement.

Le quatrième chapitre introduit la technique de prolongement lisse. Nous l'appliquons à des problèmes elliptiques en dimension 1 et 2.

Le cinquième chapitre décrit plusieurs simulations numériques. Nous testons l'efficacité de notre nouvelle méthode sur des exemples différents et enfin obtenons une simulation du problème de Stefan à deux phases avec conditions aux limites de Gibbs-Thomson.

L'annexe rassemble quelques résultats techniques.

Chapter 1

Analyses multirésolution, ondelettes et opérateurs

Ce chapitre fournit les ingrédients de base liés à l'approximation par ondelettes pour notre problème. Nous commencons par des définitions et deux exemples d'analyses multirésolutions de $L^2(\mathbb{R})$, $L^2([0,1])$ et $L^2(\mathbb{R}^n)$. La prise en compte de conditions aux limites sur une courbe γ conduit à construire une multirésolution de $L^2(\gamma)$ qui est obtenue par transformation d'une analyse multirésolution de $L^2([0,1])$. Nous rappelons les résultats importants liés aux propriétés d'approximation et de régularité des ondelettes. Enfin, nous donnons un bref apercu des méthodes de Petrov-Galerkin/ondelettes. Nous fournissons enfin les algorithmes classiques liés à la transformation en ondelettes, par exemple l'algorithme de décomposition, l'algorithme de reconstruction, l'algorithme d'interpolation et la transformation biorthogonale obtenue après application d'un opérateur elliptique.

This chapter provides the basic ingredients related to the approximation using wavelets for our problem. We start with the definition of a multiresolution analysis of $L^2(\mathbb{R})$ and $L^2([0,1])$. Then we deduce multiresolution of $L^2(\mathbb{R}^n)$ obtained by tensor product. For the boundary value problem in 2D with boundary γ, one also needs to construct the multiresolution of $L^2(\gamma)$ that is obtained by mapping from a multiresolution of $L^2([0,1])$. We recall the important results related to wavelet approximation and regularity properties. Finally, we give a brief review of Petrov-Galerkin wavelet methods and provide some relevant algorithms related to wavelet transform, for example the decomposition algorithms, reconstruction algorithms, interpolation algorithm and biorthogonal transform obtained after application of an operator.

1.1 Analyses multirésolution de $L^2(\mathbb{R})$

1.1.1 Analyses multirésolution orthonormées de $L^2(\mathbb{R})$, fonctions d'échelle et ondelettes

Definition 1.1. *A multiresolution on $L^2(\mathbb{R})$[48] is defined as, a sequence of nested space $(V_j)_{j\in\mathbb{Z}}$, satisfying*

$$(1)\cdots V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots , \tag{1.1}$$

$$(2)\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}), \tag{1.2}$$

$$(3)\bigcap_{j\in\mathbb{Z}} V_j = \{0\}, \tag{1.3}$$

$$(4) f \in V_j \Leftrightarrow f(2^{-j}) \in V_0, \tag{1.4}$$

$$(5) f \in V_0 \Leftrightarrow f(x-n) \in V_0, n \in \mathbb{Z}. \tag{1.5}$$

(6) *There exists $\phi \in V_0$ such that $\{\phi(x-n), n \in \mathbb{Z}\}$ is an Hilbertian basis of space V_0.*

(1.6)

We call ϕ the *scaling function* of the multiresolution analysis. The multiresolution analysis is said to be of m-regularity ($m \in I\!N^*$) if and only if

$$\phi \in C^{m-1} \text{ and } \forall k \leq m, \forall r, \exists c, \text{ such that } , \frac{\partial^k \phi}{\partial x^k} \leq \frac{c}{(1+|x|^r)}. \qquad (1.7)$$

The orthonormality of the $\{\phi(\cdot - k)\}_{k\in Z\!\!Z}$ can be relaxed : one only needs to require that $\{\phi(\cdot - k)\}_{k\in Z\!\!Z}$ constitute a Riesz basis. The family $\{\phi(\cdot - k)\}_{k\in Z\!\!Z}$ forms a Riesz basis for V_0 if and only if it spans V_0 and if there exists $A > 0$, $B < \infty$ such that, for any $(c_k)_{k\in Z\!\!Z} \in l^2(Z\!\!Z)$,

$$A \sum_k | c_k |^2 \leq \| \sum_k c_k \phi(\cdot - k) \|^2 \leq B \sum_k | c_k |^2 .$$

Indeed, following [27] an orthonormal basis $\phi^\sharp(\cdot - k)$ for V_0 can be constructed from $\{\phi(\cdot - k); k \in Z\!\!Z\}$, defining

$$\hat{\phi}^\sharp(\xi) = [\sum_l | \hat{\phi}(\xi + l) |^2]^{-1/2} \hat{\phi}(\xi),$$

where $\hat{\phi}$ is the Fourier transform in $L^2(I\!R)$.

For every $j \in Z\!\!Z$, if W_j stands for the orthogonal complement of V_j in V_{j+1}, we have

$$V_{j+1} = V_j \oplus W_j,$$

and

$$W_j \perp W_{j'}, \text{if } j \neq j'.$$

This implies with (1.2) and (1.3) that

$$L^2(I\!R) = \bigoplus_{j \in Z\!\!Z} W_j.$$

The fundamental theorem from [50] is that there exists an orthonormal basis $\{\psi_{0,k}; k \in Z\!\!Z\}$ for W_0. Then $\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k); k \in Z\!\!Z\}$ is an orthonormal basis for W_j, for any $j \in Z\!\!Z$ and $\{\psi_{j,k}, j \in Z\!\!Z, k \in Z\!\!Z\}$ is an Hilbertian basis of $L^2(R)$. The function ψ is called the *wavelet* of the multiresolution analysis.

Since $\phi \in V_0 \subset V_1$, and the $\phi_{1,n}$ are an orthonormal basis in V_1, we have

$$\phi = \sum_n h_n \phi_{1,n}, \qquad (1.8)$$

with

$$h_n =< \phi, \phi_{1,n} > \quad \text{and} \quad \sum_{n\in\mathbb{Z}} \mid h_n \mid^2 = 1.$$

Formula (1.8) is called the *scaling relation* and reads after a Fourier transform

$$\hat{\phi}(\xi) = m_0(\xi/2)\hat{\phi}(\xi/2), \tag{1.9}$$

where

$$m_0(\xi) = \frac{1}{\sqrt{2}} \sum_n h_n e^{-i2\pi n\xi}.$$

Equality (1.9) holds pointwise almost everywhere, m_0 is a 1-periodic C^∞ function in $L^2(]0,1[)$, named as the *symbol* of the multireslution analysis. The orthonormality of the $\phi(\cdot - k)$ implies

$$\sum_l |\hat{\phi}(x+l)|^2 = 1,$$

and

$$\mid m_0(\zeta) \mid^2 + \mid m_0(\zeta + \frac{1}{2}) \mid^2 = 1 \quad \text{a.e.} \quad . \tag{1.10}$$

So it is natural to define

$$\hat{\phi}(\xi) = \prod_{j=1}^{\infty} m_0(2^{-j}\xi)$$

If this infinite product converges pointwise a.e., then $\hat{\phi}(\xi) \in L^2(\mathbb{R})$ and $\|\phi\|_{L^2} \leq 1$.

The wavelet is defined as,

$$\psi = \sum_n g_n \phi_{1,n}, \tag{1.11}$$

with $g_n = (-1)^n h_{-n+1}$.

Following I.Dauchebies[24], we can also start the construction from an appropriate choice for the scaling function ϕ such that

$$\phi(x) = \sum_n c_n \phi(2x - n),$$

where $\sum_n \mid c_n \mid^2 < \infty$,

$$0 < \alpha \leq \sum_{l\in\mathbb{Z}} \mid \hat{\phi}(\xi + l) \mid^2 \leq \beta < \infty,$$

$$\phi \quad \text{and} \quad \hat{\phi} \quad \text{have reasonably decay,}$$

$$\int dx\phi(x) \neq 0.$$

Indeed under such conditions, $V_j = \text{span}\{\phi_{j,k}, k \in \mathbb{Z}\}$ constitute a multiresolution analysis and (1.11) defines the associated wavelet. The scaling function ϕ has a compact support if and only if $(h_n)_n$ is of finite length, i.e., all except finite h_n are equal to 0. This fact is proved in [28].

1.1.2 Analyses multirésolution biorthonormals de $L^2(\mathbb{R})$

There does not exist orthonormal wavelet basis, except Haar basis, which is simultaneously of compact support, symmetric and with real filter $(h_n)_n$. To obtain the symmetry, one should give up the property of orthonormality and introduce the biorthonormal multiresolution. We recall the main theorem of [20]:

Theorem 1.2. *Let $(h_n)_n$, $(\tilde{h}_n)_n$ be finite real sequences satisfying*

$$\sum_n h_n \tilde{h}_{n+2k} = \delta_{k0}.$$

Define

$$m_0(\xi) = 2^{-1/2} \sum_n h_n e^{-in\xi}, \quad \tilde{m}_0(\xi) = 2^{-1/2} \sum_n \tilde{h}_n e^{-in\xi},$$

$$\hat{\phi}(\xi) = (2\pi)^{-1/2} \prod_{j=1}^{\infty} m_0(2^{-j}\xi), \quad \hat{\tilde{\phi}}(\xi) = (2\pi)^{-1/2} \prod_{j=1}^{\infty} \tilde{m}_0(2^{-j}\xi).$$

Suppose that, for some C, $\epsilon > 0$

$$\begin{aligned} \mid \hat{\phi}(\xi) \mid &\leq C(1+ \mid \xi \mid)^{-1/2-\epsilon}, \\ \mid \hat{\tilde{\phi}}(\xi) \mid &\leq C(1+ \mid \xi \mid)^{-1/2-\epsilon}. \end{aligned} \tag{1.12}$$

Define

$$\psi(x) = \sqrt{2} \sum_n (-1)^n \tilde{h}_{-n+1} \phi(2x+n), \quad \tilde{\psi}(x) = \sqrt{2} \sum_n (-1)^n h_{-n+1} \tilde{\phi}(2x+n).$$

Then for any $f \in L^2(\mathbb{R})$

$$f = \sum_{j,k \in \mathbb{Z}} < f, \tilde{\psi}_{jk} > \psi_{jk} = \sum_{j,k \in \mathbb{Z}} < f, \psi_{jk} > \tilde{\psi}_{jk},$$

where the series converge strongly.

Moreover, the ψ_{jk}, $\tilde{\psi}_{jk}$ constitute two dual Riesz bases with

$$< \psi_{jk}, \tilde{\psi}_{j'k'} >= \delta_{jj'}\delta_{kk'},$$

if and only if

$$\int dx\phi(x)\tilde{\phi}(x-k) = \delta_{k0}. \tag{1.13}$$

Remark 1.1.1. P. Ponenti [61] has generalized this theorem to infinite sequence $(h_n)_n$, $(\tilde{h}_n)_n$.

1.1.3 Deux exemples d'analyse multirésolution de $L^2(I\!R)$

Example1: Spline Multiresolution

Spline functions are of common use in the approximation of functions. They are particularly attractive because of their numerical simplicity. Firstly, we give the basic properties of spline borrowed from [17].

Theorem 1.3. *The m^{th} order cardinal B-spline N_m satisfies the following properties:*

(1) supp $N_m = [0, m]$.

(2) $N_m'(x) = (\triangle N_{m-1})(x) = N_{m-1}(x) - N_{m-1}(x-1)$.

(3) The cardinal B-splines N_m and N_{m-1} are related by the identity:

$$\begin{cases} N_1(x) = \chi_{[0,1]}, \\ N_m(x) = \dfrac{x}{m-1}N_{m-1}(x) + \dfrac{m-x}{m-1}N_{m-1}(x-1). \end{cases}$$

The Fourier transforms of m(even) order spline N_m is

$$\hat{N}_m(w) = e^{-i\pi wm}(\frac{sin(\pi w)}{\pi w})^m.$$

The $\{N_m(x-k), k \in Z\!\!Z\}$ form a Riesz basis of the space V_0 of spline of order m with matching at the integers. The corresponding V_j, $j \in Z\!\!Z$ form a multiresolution. Applying the previous construction we get

$$\hat{\phi}(w) = \frac{e^{i\pi wm}\hat{N}_m(w)}{[P_{m-1}(sin^2(\pi w))]^{1/2}},$$

$$\hat{\psi}(x) = \frac{e^{-iw\pi}}{(\pi w/2)^m}\left[\frac{P_{m-1}(cos^2(\pi w/2))}{P_{m-1}(sin^2(\pi w))P_{m-1}(sin^2(\pi w/2))}\right]^{1/2} sin^{2m}(\frac{\pi w}{2}),$$

where the polynomial P_m is given in [60](see Appendix).

Since we have

$$\frac{1}{[P_{m-1}(sin^2(\pi w))]^{1/2}} = \sum_k b_k e^{-2\pi ikw},$$

ϕ can be written a linear combination of B-spline as:

$$\phi(x) = \sum_k b_k N_m(x - k + \frac{m}{2}).$$

The orthonormal spline wavelet is constructed following formula (1.11). Their plots are shown in figure 1.1 for an order $m = 4$. The following theorem [17] provides the important properties concerning the spline multiresolution,

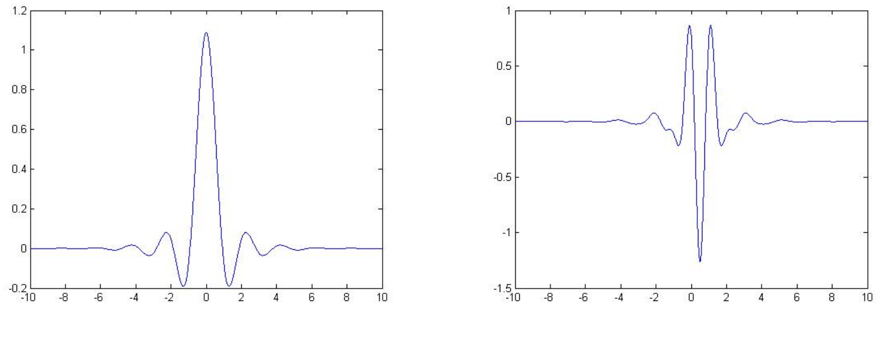
Theorem 1.4. *The functions ϕ and ψ of the orthonormal spline multiresolution satisfy:*

(1) ϕ and ψ are not of compact support but have exponential decay.

(2) ϕ and ψ are $C^{m-2}(I\!R)$.

(3) ϕ satisfies $\int_{I\!R} \phi(x) = 1$ and $\int_{I\!R} x^k \phi(x) = 0$, $k = 1, \cdots, 2m-1$.

(4) ψ has m vanishing moments, i.e. $\int_{I\!R} x^k \psi(x) = 0$, $k = 0, \cdots, m-1$.



(a) orthonormal spline scaling function (b) orthonormal spline wavelet

Figure 1.1: orthonormal spline function with order 4

<u>Example2: Compact support wavelets</u>

We will use the compactly supported orthogonal wavelets defined in[27] with N vanishing moments. The functions ϕ and ψ are of same compact support $[-N+1, N]$. All pictures in figure 1.2 are obtained by the cascade algorithm[24].

1.2 Analyses multirésolution orthonormées de $L^2([0,1])$

1.2.1 Analyse multirésolution de $L^2(T)$

The definition of the multiresolution of $L^2(T)$ is given and explicited in [60],[50] with T as $I\!R/Z\!\!Z$, that corresponds to the periodic function of period 1.

Definition 1.5. *A multiresolution analysis of $L^2(T)$ is defined by a sequence of subspace V_j, $j \geq 0$,which satisfy:*

(1) $V_0 \subset V_1 \subset \cdots \subset V_p \subset \cdots L^2(T)$.

(2) $\bigcup_{j\geq 0} V_j$ *is dense in* $L^2(T)$.

(3) $V_0 = \{$*constant functions*$\}$,

$$f(x) \in V_j \Rightarrow f(2x) \in V_{j+1},$$
$$f(x) \in V_{j+1} \Rightarrow f(\frac{x}{2}) + f(\frac{x}{2} + \frac{1}{2}) \in V_j.$$

(4) $dimV_j = 2^j$*, and there exists an orthonormal basis for* V_j, $(\phi_k^j)_{0\leq k<2^j}$

$$\phi_k^j(x) = \phi_0^j(x - \frac{k}{2^j}).$$

One obtains such a multiresolution periodizing the scaling function and wavelets of an orthonormal multiresolution on $L^2(I\!R)$ as follows

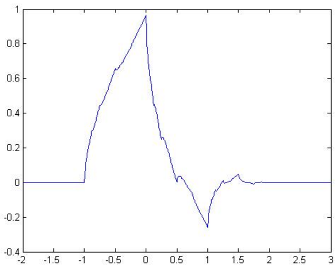
$$\phi_k^j = 2^{j/2} \sum_{z\in Z\!\!Z} \phi(2^j(x+z) - k), \quad j \geq 0, 0 \leq k < 2^j.$$

Moreover, if

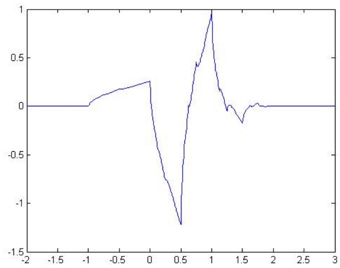
$$W_j = \text{vect}\{\psi_k^j, 0 \leq k \leq 2^j\},$$

where

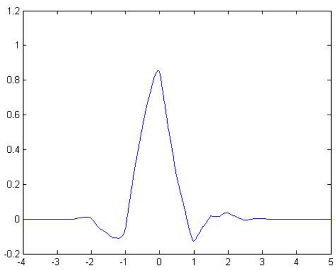
$$\psi_k^j = 2^{j/2} \sum_{z\in Z\!\!Z} \psi(2^j(x+z) - k), \quad j \geq 0, 0 \leq k < 2^j,$$



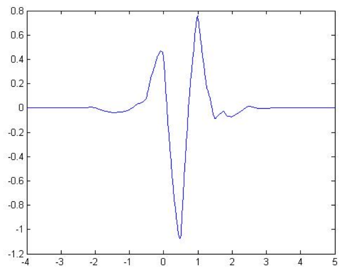
(a) scaling function with $N = 2$



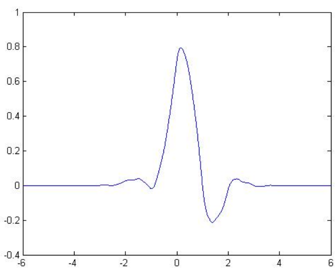
(b) wavelet with $N = 2$



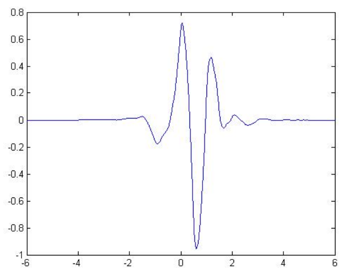
(c) scaling function with $N = 4$



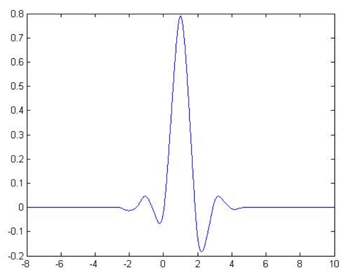
(d) wavelet with $N = 4$



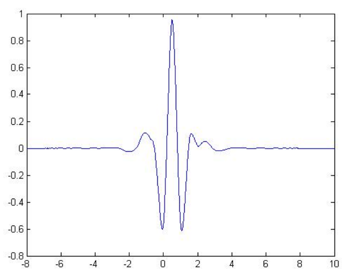
(e) scaling function with $N = 5$



(f) wavelet with $N = 5$



(g) scaling function with $N = 8$



(h) wavelet with $N = 8$

Figure 1.2: Plots of the scaling function ϕ and the wavelets ψ for the least asymmetric compactly supported wavelets with maximum number of vanishing moments

then

$$V_{j+1} = V_j \bigoplus W_j.$$

The Fourier transform of ϕ_0^j and ψ_0^j are

$$\hat{\phi}_0^j(k) = \frac{1}{2^{j/2}}\hat{\phi}(\frac{k}{2^j}),$$

$$\hat{\psi}_0^j(k) = \frac{1}{2^{j/2}}\hat{\psi}(\frac{k}{2^j}),$$

where $k \in \mathbb{Z}$.

Example: Periodic spline functions

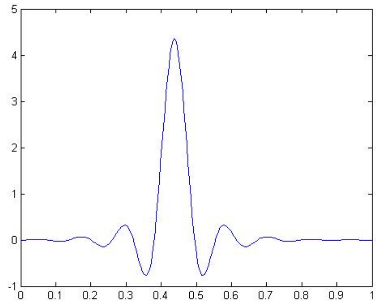
We will need any point value of $\phi_k^j(\cdot)$ to calculate its trace on an arbitrary curve γ. Since N_m is compactly supported, the periodic spline function is built from a finite linear combination of N_m, i.e.

$$\phi_0^j(x) = 2^{j/2}\sum_{l=-m/2}^{2^j+m/2} b_l N_m(2^j x - l + m/2).$$

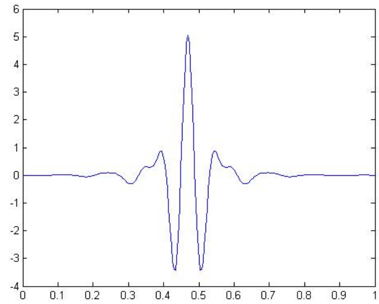
Since $\phi_k^j(\cdot)$, $0 \leq k < 2^j$ is a translation of $\phi_0^j(\cdot)$, its point values are known. The periodic spline scaling functions and wavelets of $L^2([0,1])$ with spline order 4 are seen in figure 1.3 obtained by the construction and interpolation process.

1.2.2 Analyse multirésolution de $L^2(]0,1[)$

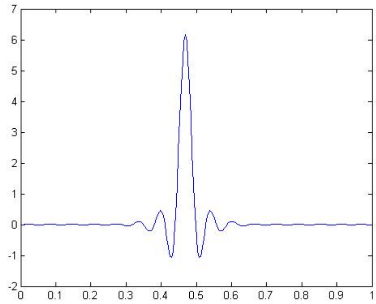
A construction to adapt a multiresolution of $L^2(\mathbb{R})$ to the interval $[0,1]$ was first introduced by Y.Meyer in [51]. Here the scaling functions are obtained by orthonormalizing the restriction to $[0,1]$ of $\phi_{j,k}$(defined on $\mathbb{R}$) which support intersect the interval. The main problem of this construction is the ill-conditioning of the orthonormalization process of the restricted scaling functions. To circumvent this disadvantage, A.Cohen et al. [21] proposed to construct the edge scaling function as specific linear combinations of $\phi_{j,k}$ while preserving the previous method's advatages: regularity, vanishing moments, spatial localization and existence of a fast algorithm. Its construction restricts to scale j satisfying $2^j \geq 2N$.



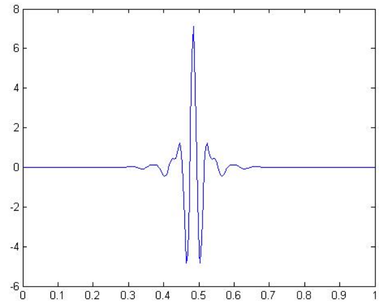
(a) periodic spline scaling function ϕ_7^4



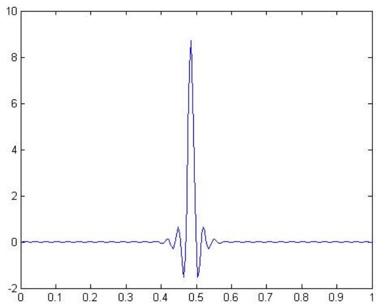
(b) periodic spline wavelet ψ_7^4



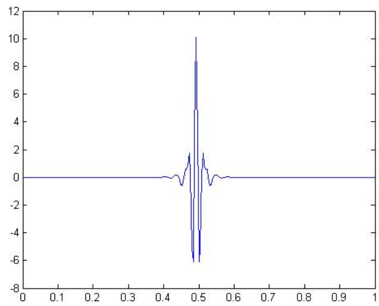
(c) periodic spline scaling function ϕ_{15}^5



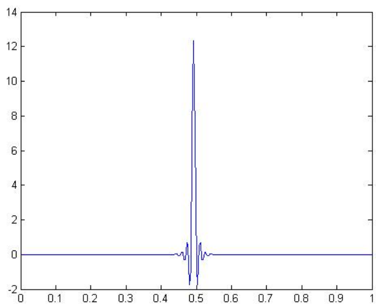
(d) periodic spline wavelet ψ_{15}^5



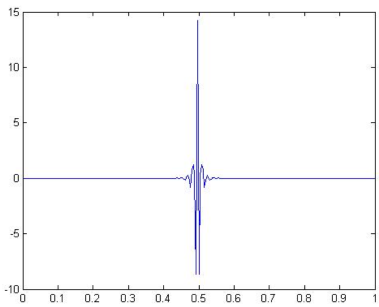
(e) periodic spline scaling function ϕ_{31}^6



(f) periodic spline wavelet ψ_{31}^6



(g) periodic spline scaling function ϕ_{63}^7



(h) periodic spline wavelet ψ_{63}^7

Figure 1.3: Plots of the periodic scaling functions and wavelets with spline order 4

The starting point of the construction [21] is the N vanishing moment family of compact support wavelet [27]. Then one retains the interior scaling functions, and adds adapted edge scaling functions in such a way that their union still generates all polynomial on $[0, 1]$, up to degree $N - 1$. The left edge scaling functions are constructed from the functions defined as,

$$\tilde{\phi}^0_{j,k} = 2^{j/2} \sum_{n=k}^{2N-2} \begin{pmatrix} n \\ k \end{pmatrix} \phi(2^j x + n - N + 1)\chi_{[0,1]}.$$

The supports $\mathrm{supp}(\tilde{\phi}^0_{j,k})$ are staggered, i.e., $\mathrm{supp}(\tilde{\phi}^0_{j,k}) = [0, \frac{2N-1-k}{2^j}]$ and $\tilde{\phi}^0_{j,k}$ is a polynomial of degree k on the interval $[0, \frac{1}{2^j}]$. The left edge scaling functions $\phi^0_{j,k}$, $k = 0, \cdots, N - 1$ are obtained from orthonormalizing and reordering of $\tilde{\phi}^0_{j,k}$. The right edge scaling functions derive from

$$\phi^1_{j,k}(x) = \phi^{0,\sharp}_{j,k}(1 - x), \quad k = 0, \cdots, N - 1,$$

where $\phi^{0,\sharp}_{j,k}$ is the left edge scaling function constructed with filter coefficient $h^\sharp_n = h_{-n+1}$. For wavelets, one first defines,

$$\tilde{\psi}^k = \phi^0_{1,k} - \sum_{m=0}^{N-1} (\phi^0_{1,k}, \phi^0_{0,m})\phi^0_{0,m}.$$

One replaces the family $\tilde{\psi}^k$ by $\tilde{\tilde{\psi}}^k$, which are the linear combination of the $\tilde{\psi}^k$, so that their supports are staggered. After orthonormalization, they produce the left edge wavelets, the right ones are obtained in the same way as for the right edge scaling functions. Finally, for $2^J > 2N$, The collection

$$\begin{aligned} &\bigcup_{j\geq J} [\{\psi^0_{j,k}, k = 0, \cdots, N - 1\} \cup \{\psi_{j,m}, m = N, \cdots, 2^j - N - 1\} \\ &\quad \cup \{\psi^1_{j,k}, k = 0, \cdots, N - 1\}] \\ &\quad \cup \{\phi^0_{J,k}, k = 0, \cdots, N - 1\} \cup \{\phi_{J,m}, m = N, \cdots, 2^J - N - 1\} \\ &\quad \cup \{\phi^1_{J,k}, k = 0, \cdots, N - 1\}, \end{aligned}$$

is an orthonormal basis of $L^2([0, 1])$. The scaling relations at the edge read:

$$\phi^0_{j,k}(x) = \sum_{l=0}^{N-1} H^0_{k,l}\phi^0_{j+1,l}(x) + \sum_{m=N}^{N+2k} H^0_{k,m}\phi_{j+1,m}(x), \quad k = 0, \cdots, N - 1, \qquad (1.14)$$

$$\psi^0_{j,k}(x) = \sum_{l=0}^{N-1} G^0_{k,l}\phi^0_{j+1,l}(x) + \sum_{m=N}^{N+2k} G^0_{k,m}\phi_{j+1,m}(x), \quad k = 0, \cdots, N-1,$$

$$\phi^1_{j,r}(x) = \sum_{l=0}^{N-1} H^{0,\sharp}_{r,l}\phi^{0,\sharp}_{j+1,l}(1-x) + \sum_{m=N}^{N+2r} H^{0,\sharp}_{r,m}\phi^{\sharp}_{j+1,m}(1-x)$$
$$r = 0, \cdots, N-1,$$

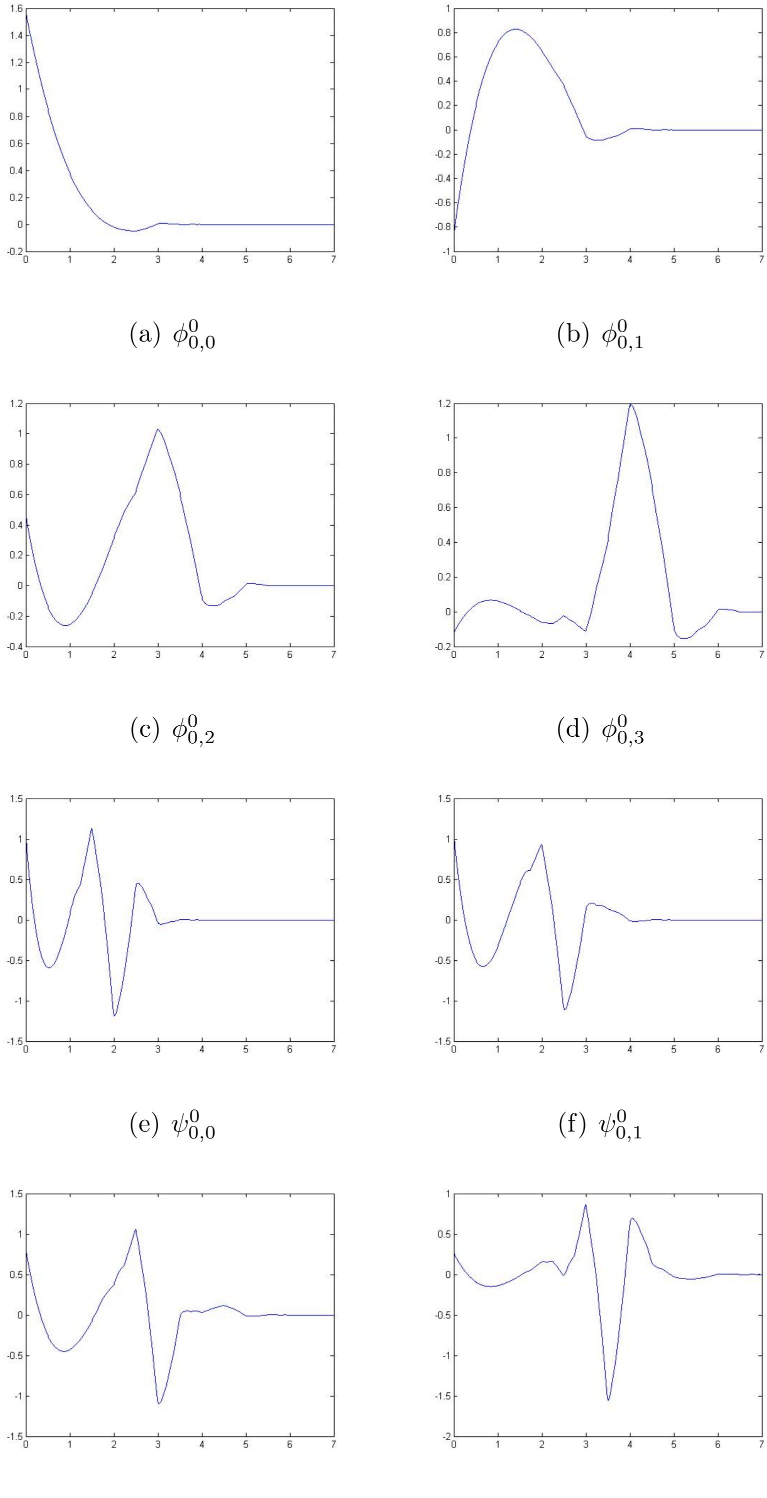
$$\psi^1_{j,r}(x) = \sum_{l=0}^{N-1} G^{0,\sharp}_{r,l}\phi^{0,\sharp}_{j+1,l}(1-x) + \sum_{m=N}^{N+2r} G^{0,\sharp}_{r,m}\phi^{\sharp}_{j+1,m}(1-x)$$
$$r = 0, \cdots, N-1.$$

Plots of scaling function and wavelet obtained by the cascade algorithm with $N = 4$ are available in figure 1.4 and 1.5. In [15][53], another family of wavelets on the interval satisfying homogeneous boundary conditions is proposed. Its construction is mainly based on the fact that the left and right edge scaling function $\phi^0_{j,k}$, $\phi^1_{j,k}$ are polynomial of degree $N-1$ on the interval $[0, \frac{1}{2^j}]$ and $[1-\frac{1}{2^j}, 1]$ respectively.

As a generalization of this classical construction of compactly supported wavelet on the interval $[0,1]$, we recall the work of S. Bertoluzze et al [10]. Their approach overcomes the restriction on the scale j that guarantees non interacting boundaries and build wavelet on large scale. This construction coincides with the classical one when $j \geq log_2 2N$. First, the subspace V^*_j for $j \geq log_2(2(N-M)-1)$(M the order of polynomial reproduction of ϕ(see section 1.5)) which is of $N_j = 2^j - 2N + 2M + 3$ dimension and $M_j = \min\{M, N_j - 1\}$ polynomial reproduction is introduced. More precisely,

$$\begin{aligned} V^*_j :=& \{f = \sum_{k\in\mathbb{Z}} f^j_k \phi_{j,k}, \exists p_l, p_r \in P_{M_j} \text{ such that} \\ & f^j_k = p_l(k), \forall k \in]-\infty, N-1], \\ & f^j_k = p_r(k), \forall k \in [2^j - N + 1, +\infty[, \end{aligned}$$

where P_M is the space of polynomial of degree less than or equal to M. Then $V_j[0,1]$, a multiresolution of $L^2[0,1]$, is defined by taking the restriction to $[0,1]$.



(a) $\phi^0_{0,0}$ (b) $\phi^0_{0,1}$

(c) $\phi^0_{0,2}$ (d) $\phi^0_{0,3}$

(e) $\psi^0_{0,0}$ (f) $\psi^0_{0,1}$

(g) $\psi^0_{0,2}$ (h) $\psi^0_{0,3}$

Figure 1.4: Left edge interval scaling functions and wavelets (Daubechies $N = 4$)

$\{\check{\phi}_{j,k}[0,1]\}_{k\in I_j}$ can be a Riesz basis for $V_j[0,1]$, obtained by an extension operator E_j from $S(I_j)$ to $S(\mathbb{Z})(S(\mathcal{I})$ denoting the space of real valued sequences with indexes in the set $\mathcal{I})$, $I_j = \{N - M - 1, \cdots, 2^j - N + M + 1\}$. E_j extends the sequence of $S(I_j)$ on the left and on the right by Lagrange polynomial of degree M_j.

$$\check{\phi}_{j,k}[0,1] = \sum_{m=-N}^{N+2^j} \eta_m^{j,k} \phi j, m|_{[0,1]} \text{ with } \eta_m^{j,k} = E_j \delta^{j,k}$$

where $\delta^{j,k} = (\delta_n^{j,k})_{n\in I_j} \in S(I_j)$,

$$\delta_n^{j,k} = \begin{cases} 0, n \neq k \\ 1, n = k. \end{cases}$$

Figure 1.6 shows the Riesz basis $\check{\phi}_{j,k}[0,1]$ by Daubechies wavelet.

1.3 Analyse multirésolution de $L^2(\gamma)$

When using wavelet bases to solve the boundary value problem with the boundary γ in 2D, it is necessary to construct a wavelet basis on γ where γ is piecewise regular. Writing $\gamma = \bigcup_{k\in K} \gamma_k$, if H_k denotes the transform such that

$$\gamma_k = H_k([0,1]),$$

ϕ_0^γ is defined by

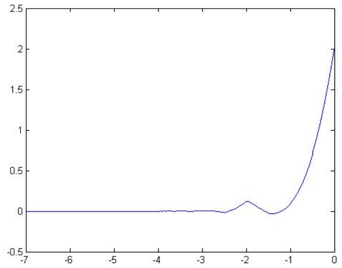
$$\forall s \in \gamma_k, \quad \phi_0^\gamma(s) = \phi_0^{[0,1]}(H_k^{-1}(s)),$$

where $\phi_0^{[0,1]}$ is the scaling function of the multiresolution on the interval. The orthonormality of the famililY $\{\phi_\alpha^\gamma\}_\alpha$ does not hold in the classical L^2-sense. The suitable scalar product in L_H^2 involves the Jacobian J_H of the transformation as

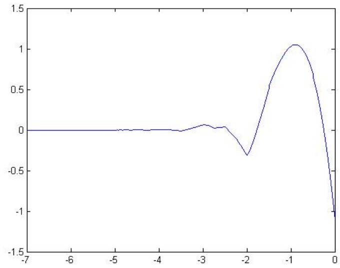
$$\forall (f,g) \in L^2(\gamma), \quad < f, g >_{L_H^2} = \int_\gamma f(s)g(s) \mid J_H^{-1} \mid ds.$$

1.4 Analyse multirésolution de $L^2(\mathbb{R}^n)$

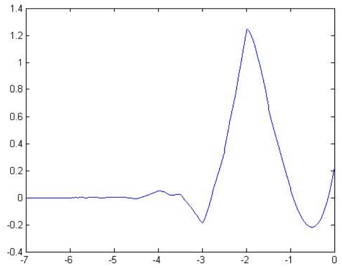
The simplest way to build multivariate wavelets is to use tensor products of univariate functions. Denote the scaling function $\Phi_{j,k}$, $k \in \mathbb{Z}^n$, the wavelet $\Psi_{j,k}^e$, $k \in \mathbb{Z}^n$,



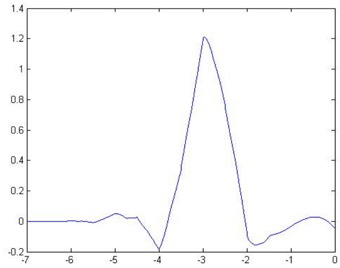
(a) $\phi^1_{0,0}$



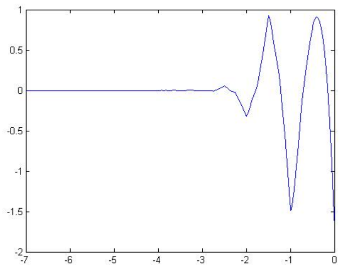
(b) $\phi^1_{0,1}$



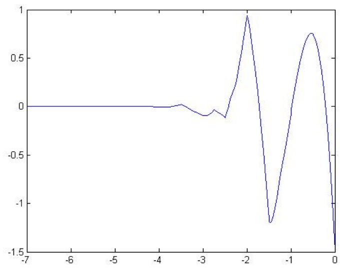
(c) $\phi^1_{0,2}$



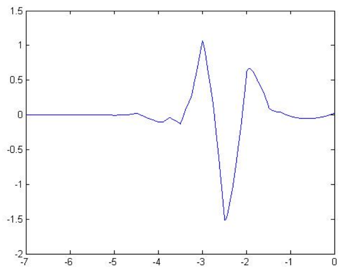
(d) $\phi^1_{0,3}$



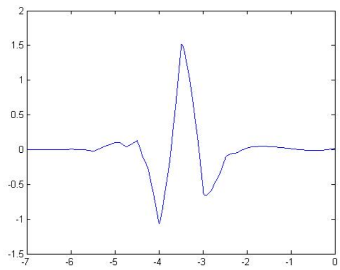
(e) $\psi^1_{0,0}$



(f) $\psi^1_{0,1}$



(g) $\psi^1_{0,2}$



(h) $\psi^1_{0,3}$

Figure 1.5: Right edge interval scaling functions and wavelets (Daubechies $N = 4$)

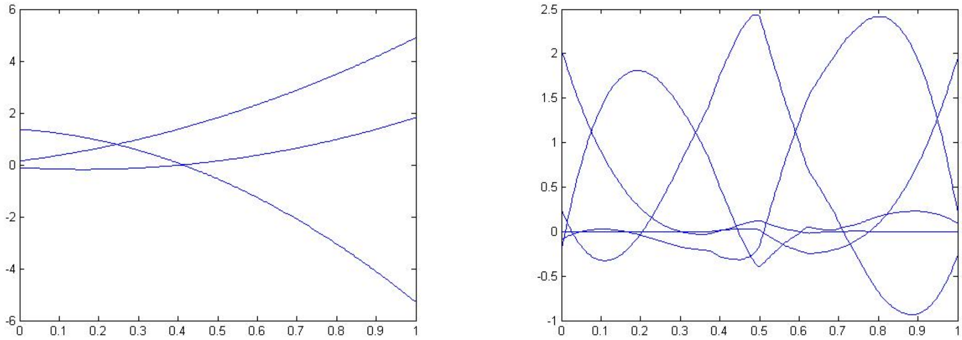


Figure 1.6: Scaling functions at large scale: Left with N=3, j=1; Right with N=4, j=2.

$e \in E = \{\{0,1\}^n \setminus (0, \cdots, 0)\}$. We take $n = 2$ for example, since higher dimensions are analogous. The approximation space V_j of $L^2(\mathbb{R}^2)$ is the tensor product space $V_j^x \otimes V_j^y$, where V_j^x, resp.V_j^y, represent the univariate space, in the variable x, resp.y. The orthonormal basis of V_j is

$$\Phi_{j;n_1,n_2}(x,y) = \phi_{j,n_1}(x)\phi_{j,n_2}(y).$$

Writing

$$\begin{aligned} V_{j+1} &= V_{j+1}^x \otimes V_{j+1}^y = (V_j^x \oplus W_j^x) \otimes (V_j^y \oplus W_j^y) \\ &= (V_j^x \otimes V_j^y) \oplus [(V_j^x \otimes W_j^y) \oplus (W_j^x \otimes V_j^y) \oplus (W_j^x \otimes W_j^y)] \\ &= V_j \oplus W_j, \end{aligned}$$

this leads to define tree wavelets,

$$\begin{aligned} \Psi^h(x,y) &= \phi(x)\psi(y), \\ \Psi^v(x,y) &= \psi(x)\phi(y), \\ \Psi^d(x,y) &= \psi(x)\psi(y). \end{aligned}$$

Then

$$\{\Psi^e_{j;n_1,n_2}; (n_1,n_2) \in \mathbb{Z}^2, e \in \{h,v,d\}\},$$

is an orthonormal basis of W_j.

1.5 Approximation et propriétés de régularité

1.5.1 Reproduction polynomiale

A multiresolution is said to be of order d if it reproduces polynomials of degree up to $d-1$. The following theorem in [19] gives a necessary and sufficient condition:

Theorem 1.6. *Let ϕ be an L^1 function with compact support with $\int \phi = 1$. The following properties are equivalent:*

(1) ϕ satisfies the Strang-Fix conditions *of order $d-1$.*

$$(\frac{\partial}{\partial w})^q \hat{\phi}(2n\pi) = 0, \quad n \in \mathbb{Z} - 0, |q| \neq d-1. \tag{1.15}$$

(2) For all $q = 0, \cdots, d-1$, one can expand the polynomial x^q according to

$$x^q = \sum_{k \in \mathbb{Z}} [k^q + P_{q-1}(k)]\phi(x-k),$$

where P_{q-1} is a polynomial of degree $q-1$, or equivalently, for all $q = 0, \cdots, d-1$

$$\sum_{k \in \mathbb{Z}} k^q \phi(x-k) = x^q + r_{q-1}(x),$$

where r_{q-1} is a polynomial of degree $q-1$.

Remark 1.5.1. Another sufficient condition which guarantee the $d-1$ order polynomial reproduction reads for the symbol m_0:

$$\partial^k m_0(\pi) = 0, \quad k = 0, \cdots, d-1.$$

1.5.2 Estimations de type Jackson et Bernstein

The following two theorems [23] relate the approximation property of the space V_j to the multiresolution analysis order d and the regularity order m. Let $\Omega \subset \mathbb{R}^n$

Theorem 1.7. *(Jackson inequality) $\forall f \in H^s(\Omega)$, $0 < s \leq d$,*

$$\|f - P_{V_j} f\|_{L^2(\Omega)} \leq C_{J,\phi} 2^{-js} \|f\|_{H^s(\Omega)},$$

where P_{V_j} is the projection on V_j, $C_{J,\phi}$ depends only on the chosen multiresolution analysis.

Theorem 1.8. *(Bernstein inequality) If* $f \in vect\{\Psi^e_{j,k}, k \in \mathbb{Z}^n, e \in E\}$*,*

$$\forall s \leq m, \quad f \in H^s(\Omega) \Rightarrow \|f\|_{H^s(\Omega)} \sim 2^{sj}\|f\|_{L^2(\Omega)}.$$

If $f \in vect\{\Phi_{j,k}, k \in \mathbb{Z}^n\}$*,*

$$\forall s < 0, \quad f \in H^s(\Omega) \Rightarrow \|f\|_{H^s(\Omega)} \geq C_{B,\phi}2^{sj}\|f\|_{L^2(\Omega)},$$

$$\forall s > 0, \quad f \in H^s(\Omega) \Rightarrow \|f\|_{H^s(\Omega)} \leq C_{B,\phi}2^{sj}\|f\|_{L^2(\Omega)},$$

where $C_{B,\phi}$ *depends only on the chosen multiresolution analysis.*

If the Jackson inequality and Bernstein inequalities hold, we have the following result proven in W. Dahmen [22]

Theorem 1.9. *(Norm equivalence)* $\forall r \leq m$ *, we have*

$$\|\sum_{\mu\in M} c_\mu\Phi_\mu + \sum_{\lambda\in\Lambda} d_\lambda\Psi_\lambda\|_{r,\Omega} \approx (\sum_{\mu\in M}|c_\mu|^2 + \sum_{\lambda\in\Lambda}2^{2jr}|d_\lambda|^2)^{1/2}$$

where $M = \{(j_0, k), k \in \mathbb{Z}^n\}$*,* $\Lambda = \{(j, k, e), j \geq j_0, k \in \mathbb{Z}^n, e \in E\}$*.*

1.6 Action des opérateurs sur un analyse multirésolution

Let P be a linear differential operator of order s defined from $H^s(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. Then P is a polynomial in the derivative D, which in multi-index notation can be written

$$P(x, D) = \sum_{|\alpha|\leq s} a_\alpha(x)D^\alpha,$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $D^\alpha = \frac{\partial^{\alpha_1}}{(i\partial x_1)^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{(i\partial x_n)^{\alpha_n}}$. Its symbol reads

$$\sigma(x, \zeta) = \sum_{|\alpha|\leq s} a_\alpha(x)\zeta^\alpha,$$

with $\zeta^\alpha = \zeta_1^{\alpha_1} \cdots \zeta_n^{\alpha_n}$. We say that P is elliptic if there exists a constant $c > 0$ such that

$$< Pv, v >_{L^2(\mathbb{R}^n)} \geq c\|v\|_{H^s(\mathbb{R}^n)}, \quad \forall v \in H^s(\mathbb{R}^n).$$

We are now going to describe the action of a differential operator on a multiresolution analysis. We recall the following theorems from [61],

Theorem 1.10. *Let ϕ be the scaling function and the wavelet ψ of regularity r and with m vanishing moments.*

Let Ψ^i, $i = 1, \cdots, 2^n - 1$ be the $2^n - 1$ orthonormal wavelets of a multiresolution of $L^2(\mathbb{R}^n)$, constructed from ϕ and ψ by tensor product method.

If L is an elliptic operator of order $s > 0$, such that

$$r \geq s + 1,$$
$$m \geq s + 1,$$

If we define

$$\tilde{\Theta}^i = 2^{-js} 2^{jn/2} L^*[\Psi^i(2^j x - k)],$$
$$\Theta^i = 2^{js} 2^{jn/2} L^{-1}[\Psi^i(2^j x - k)],$$

and then we have

(1) $\tilde{\Theta}^i \in H^{r-s}(\mathbb{R}^n)$ and $\Theta^i \in H^{r+s}(\mathbb{R}^n)$,

(2) we have following estimates:

$$|\partial^{r'} \tilde{\Theta}^i(x)| \leq C_{r'} 2^{j|r'|} 2^{jn/2} (1 + 2^j |x - \lambda|)^{-d}$$
$$|\partial^{r} \Theta^i(x)| \leq C_{r} 2^{j|r|} 2^{jn/2} (1 + 2^j |x - \lambda|)^{-n+s-m-1-|r|}$$

where $\lambda = (k + \frac{1}{2} i) 2^{-j}$, the multi-index r', r satisfying $|r'| \leq r - s$ and $|r| \leq r + s - 1$, $\forall d \in \mathbb{N}$.

(3) The functions $\tilde{\Theta}^i$ and Θ^i have some vanishing moments. More precisely, if $x^k = x_1^{k_1} \cdots x_d^{k_d}$, one has

$$\int_{\mathbb{R}} x^k \tilde{\Theta}^i(x) = 0, \quad 0 \leq |k| \leq m + s$$
$$\int_{\mathbb{R}} x^k \Theta^i = 0, \quad 0 \leq |k| \leq m - s$$

The following theorem describes how an homogeneous operator and a suitable multiresolution generate a bi-orthogonal multiresolution.

Theorem 1.11. *Let L be a homogenous elliptic operator with constant coefficients of order s and symbol σ. $(V_j)_{j \in \mathbb{Z}}$ is a multiresolution analysis of $L^2(\mathbb{R})$ of regularity r and vanishing moment m. Note ϕ the scaling function.*

If

$$m > s + \frac{n}{2},$$
$$r > s + \frac{n}{2},$$

and if S is 2π periodic function equivalent to σ around zero, then the functions defined by

$$\hat{\tau}(w) = \frac{\bar{\sigma}(w)}{\bar{S}(w)}\hat{\phi}(w)$$
$$\hat{\tilde{\tau}}(w) = \frac{S(w)}{\sigma(w)}\hat{\phi}(w)$$

are the scaling functions of a bi-orthogonal multiresolution.

1.7 Méthode d'ondelettes/Petrov-Galerkin

We consider a boundary value problem,

$$\begin{cases} Lu = f & \text{in} \quad \omega, \\ u = g & \text{on} \quad \gamma, \end{cases} \tag{1.16}$$

where ω is an open, bounded subset of R^n, f and g are given function, γ is the boundary of ω. A weak form reads:

$$\text{find} \quad u \in U \quad \text{such that} \quad A(u, v) = F(v), \quad \forall v \in V,$$

where U is a space of admissible solutions and V is a space of test functions. Both U and V will be assumed to be Hilbert spaces. In most case U is same as V. F is a linear functional on V and $A(\cdot, \cdot)$ is a bilinear form corresponding to the operator L. Classically one introduces $\tilde{g}$ a lifting of g on ω and look for the solution u as $u = \tilde{u} + \tilde{g}$. Then $\tilde{u}$ is the solution of the homogeneous Dirichlet problem

$$\begin{cases} L\tilde{u} = f - L\tilde{g} & \text{in} \quad \omega, \\ \tilde{u} = 0 & \text{on} \quad \gamma. \end{cases} \tag{1.17}$$

For example, if L be a second order linear, elliptic operator,

$$Lu := -\sum_{i,j=1}^{n}(a_{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u,$$

then the problem becomes

$$\text{find} \quad \tilde{u} \in H_0^1(\omega) \quad \text{such that} \quad A(\tilde{u}, v) = F(v), \quad \forall v \in H_0^1(\omega),$$

with

$$A(\tilde{u}, v) := \int_\omega \sum_{i,j=1}^{n} a_{ij}(x)\tilde{u}_{x_i} v_{x_j} + \sum_{i=1}^{n} b_i(x)\tilde{u}_{x_i} v + c(x)\tilde{u}v, \tag{1.18}$$

$$F(v) := \int_\omega (f - L\tilde{g})v dx.$$

In a Petrov-Galerkin method, the approximation reads

$$\text{find} \quad u_h \in U_h \quad \text{such that} \quad A_h(u_h, v_h) = F_h(v_h), \quad \forall v \in V_h, \tag{1.19}$$

where U_h, V_h are multiresolution spaces of finite dimension, $U_h \subset U$, $V_h \subset V$, $U_h \neq V_h$, dim U_h =dim V_h=N_h, for all $h > 0$. $A_h : U_h \times V_h \to I\!R$ and $F_h : V_h \to I\!R$ are restrictions of A and F, respectively.

If $\{\Psi_\alpha^\lambda, \alpha = (j, k_1, \cdots, k_n) \in Z\!\!Z^{n+1}, \lambda \in E\}$ is a basis of U_h and $\{\tilde{\Psi}_\beta^\rho, \beta = (j, k_1, \cdots, k_n) \in Z\!\!Z^{n+1}, \rho \in E\}$ a basis of V_h with $E = \{(0,1)\}^n \setminus (0, \cdots, 0)$, writing

$$u_h = \sum_{\alpha,\lambda} \xi_\alpha^\lambda \Psi_\alpha^\lambda,$$

equation (1.19) then reduces to the following linear system

$$\mathcal{A}\xi = \mathcal{F},$$

with $\xi = (\xi_\alpha^\lambda)$, $\mathcal{F} = (F_h(\tilde{\Psi}_\beta^\rho))$, $\mathcal{A} = A_h(\Psi_\alpha^\lambda, \tilde{\Psi}_\beta^\rho)$. The matrix $\mathcal{A}$ is called the *stiffness matrix.*

For the analysis of stability and convergence of (1.19), we have the following theorem [63]

Theorem 1.12. *Let U and V be two Hilbert space, $F_h : V_h \to I\!R$ be a linear map, $A_h : U_h \times V_h \to I\!R$ be a bilinear form such that there exists two positive constants γ_h and α_h satisfying*

$$|A_h(u_h, v_h)| \leq \gamma_h \|u_h\|_U \|v_h\|_V, \quad \forall u_h \in U_h, v_h \in V_h,$$

$$\sup_{\substack{v_h \in V_h \\ v_h \neq 0}} \frac{A_h(u_h, v_h)}{\|v_h\|_V} \geq \alpha_h \|u_h\|_U, \quad \forall u_h \in U_h,$$

$$\sup_{u_h \in U_h} A_h(u_h, v_h) > 0, \quad \forall v_h \in V_h, v_h \neq 0.$$

Then, there exists a unique solution u_h to (1.19) that satisfies

$$\|u_h\|_U \leq \frac{1}{\alpha_h} \sup_{\substack{v_h \in V_h \\ v_h \neq 0}} \frac{F_h(v_h)}{\|v_h\|_V}.$$

Remark 1.7.1. In chapter 3 we will use appropriate choices for Ψ and $\tilde{\Psi}$ such that the stiffness matrix reduces to the identity matrix.

1.8 Algorithmes numériques standard

1.8.1 Algorithmes de décomposition

Decomposition on $L^2(I\!R)$

If $c_k^j = (f, \phi_{j,k})$, $d_k^j = (f, \psi_{j,k})$, then from the relations (1.8) and (1.11), one gets the tree algorithm that reads:

$$c_k^{j-1} = \sum_n \overline{h_{n-2k}} c_n^j,$$

$$d_k^{j-1} = \sum_n \overline{g_{n-2k}} c_n^j.$$

Decomposition on $L^2(T)$

From [60], there exists two 2^j-periodic filters $\{H_j(n)\}_n$ and $\{G_j(n)\}_n$ such that

$$c_k^{j-1} = \sum_{i=0}^{2^j-1} c_i^j H_j(i-2k),$$

$$d_k^{j-1} = \sum_{i=0}^{2^j-1} c_i^j G_j(i-2k),$$

where $0 \leq k < 2^{j-1}$. Moreover, H_j and G_j satisfy:

$$\hat{H}_j(k) = \frac{1}{2^j} H(\frac{k\pi}{2^j}),$$

and

$$\hat{G}_j(k) = \frac{1}{2^j} G(\frac{k\pi}{2^j}).$$

For the spline multiresolution,

$$H(x) = \sqrt{2} \cos^m x [\frac{P_{m-1}(\sin^2 x)}{P_{m-1}(\sin^2(2x))}]^{1/2},$$

and

$$G(x) = \sqrt{2} e^{-2ix} \sin^m x [\frac{P_{m-1}(\cos^2 x)}{P_{m-1}(\sin^2(2x))}]^{1/2}.$$

Decomposition on $L^2([0,1])$

$$c_k^{j-1} = \sum_{l=0}^{N-1} H_{k,l}^0 c_l^j + \sum_{m=N}^{N+2k} H_{k,m}^0 c_m^j, \quad k = 0, \cdots, N-1,$$

$$c_k^{j-1} = \sum_{m=2k-N+1}^{2k+N} h_{m-2N} c_m^j, \quad k = N, \cdots, 2^{j-1} - N - 1,$$

$$c_k^{j-1} = \sum_{l=0}^{N-1} H_{2^{j-1}-1-k,l}^{0,\sharp} c_{2^j-1-l}^j + \sum_{m=N}^{N+2(2^{j-1}-1-k)} H_{2^{j-1}-1-k,m}^{0,\sharp} c_{2^j-1-m}^j$$
$$k = 2^{j-1} - N, \cdots, 2^{j-1} - 1,$$

$$d_k^{j-1} = \sum_{l=0}^{N-1} G_{k,l}^0 c_l^j + \sum_{m=N}^{N+2k} G_{k,m}^0 c_m^j, \quad k = 0, \cdots, N-1,$$

$$d_k^{j-1} = \sum_{m=2k-N+1}^{2k+N} g_{m-2N} c_m^j, \quad k = N, \cdots, 2^{j-1} - N - 1,$$

$$d_k^{j-1} = \sum_{l=0}^{N-1} G_{2^{j-1}-1-k,l}^{0,\sharp} c_{2^j-1-l}^j + \sum_{m=N}^{N+2(2^{j-1}-1-k)} G_{2^{j-1}-1-k,m}^{0,\sharp} c_{2^j-1-m}^j$$
$$k = 2^{j-1} - N, \cdots, 2^{j-1} - 1.$$

1.8.2 Algorithmes de reconstruction

Reconstruction on $L^2(\mathbb{R})$

$$c_n^j = \sum_k [h_{n-2k} c_k^{j-1} + g_{n-2k} d_k^{j-1}].$$

Reconstruction on $L^2(T)$

$$c_n^j = \sum_{k=0}^{2^{j-1}-1} c_k^{j-1} H_j(n-2k) + \sum_{k=0}^{2^{j-1}-1} d_k^{j-1} G_j(n-2k).$$

Reconstruction on $L^2([0,1])$

$$c_k^j = \sum_{n=0}^{N-1} c_n^{j-1} H_{n,k}^0 + \sum_{n=0}^{N-1} d_n^{j-1} G_{n,k}^0, \quad k = 0, \cdots, N-1,$$

$$\begin{aligned} c_k^j = & \sum_{n_1=0}^{N-1} c_n^{j-1} A_1 + \sum_{n_2=N}^{2^{j-1}-N-1} c_n^{j-1} A_2 + \sum_{n_3=2^{j-1}-N}^{2^{j-1}-1} c_n^{j-1} A_3 \\ & \sum_{n_1=0}^{N-1} d_n^{j-1} B_1 + \sum_{n_2=N}^{2^{j-1}-N-1} d_n^{j-1} B_2 + \sum_{n_3=2^{j-1}-N}^{2^{j-1}-1} d_n^{j-1} B_3, \quad k = N, \cdots, 2^j - N - 1, \end{aligned}$$

where $A_1 = H_{n,k}^0$, $B_1 = G_{n,k}^0$ if $k \leq N + 2n_1$; $A_2 = h_{k-2n}$, $B_2 = g_{k-2n}$ if $2n_2 - N + 1 \leq k \leq 2n_2 + N$; $A_3 = H^{0,\sharp}_{2^{j-1}-1-n_3, 2^j-1-k}$, $B_3 = G^{0,\sharp}_{2^{j-1}-1-n_3, 2^j-1-k}$ if $k \geq 1 + 2n_3 - N$.

$$c_k^j = \sum_{n=2^{j-1}-N}^{2^{j-1}-1} c_n^{j-1} H^{0,\sharp}_{2^{j-1}-1-n, 2^j-1-k} + \sum_{n=2^{j-1}-N}^{2^{j-1}-1} d_n^{j-1} G^{0,\sharp}_{2^{j-1}-1-n, 2^j-1-k}$$
$$k = 2^j - N, \cdots, 2^j - 1.$$

1.8.3 Processus d'interpolation

If there exists a function $S \in V_j$ satisfying:

$$S(\frac{k}{2^j}) = \delta_{0,k}, k = 0, \cdots, 2^j - 1,$$

then there exists an interpolation operator in V_j that reads

$$< \tilde{f}_j, \phi_k^j >= \sum_{i=0}^{2^j-1} f_i L_j(k-i).$$

The function $\tilde{f}_j$ satisfies

$$\tilde{f}_j(k/2^j) = f(k/2^j) := f_k, k = 0, \cdots, 2^j - 1$$

For $L^2(T)$ and spline of even order m, we have

$$\hat{L}_j(k) = \frac{1}{2^{3j/2}} L(\frac{k\pi}{2^j}),$$

$$L(x) = \frac{[P_{m-1}(\sin^2 x)]^{1/2}}{P_{(m/2)-1}(\sin^2 x)}.$$

1.8.4 Calcul des coefficients d'échelle associés à $L^{-1}\phi$

In the next chapters, to apply the Petrov-Galerkin method making the test space adaptive to the operator L in the form $L = I + L'$ where L' is a homogenous elliptic operator, we will need to know compute scalar products with the family $L^{-1}\Phi_\alpha$. From [16], we know that $\forall \epsilon > 0$, there exists r, such that $(L)^{-1}\Phi_\alpha \in_\epsilon V_{j+r}$, where $f \in_\epsilon V_{j+r}$ means that, $\exists g \in V_{j+r}$ such that $\|f - g\|_{L^2(\mathbb{R}^n)} \leq \epsilon$. In two dimension, to get the point value of $\theta_\alpha = (L)^{-1}\Phi_\alpha$, where Φ_α is the periodic spline scaling function on domain $[0,1] \times [0,1]$, we write

$$\theta_\alpha \sim \sum_{\alpha'} C_{\alpha'}^{\alpha} \Phi_{\alpha'},$$

with $\alpha = (j; n_1, n_2)$, $0 \leq n_1, n_2 \leq 2^j - 1$, $\alpha' = (j'; k_1, k_2)$, $0 \leq k_1, k_2 \leq 2^{j'} - 1$, $j' = j + r$. Using Fourier transform, it becomes

$$\hat{\theta}_\alpha(w_1, w_2) \sim m_{j'}(\theta_\alpha)(w_1, w_2)\hat{\Phi}_{j';0,0}(w_1, w_2),$$

with

$$\begin{aligned} m_{j'}(\theta_\alpha)(w_1, w_2) &= \sum_{k_1,k_2=0}^{2^{j'}-1} C_{j';k_1,k_2}^{\alpha} e^{-i2\pi(k_1w_1+k_2w_2)/2^{j'}} \\ &= 4^{j'} \sum_{z_1,z_2\in\mathbb{Z}} \hat{\theta}_\alpha(w_1 + 2^{j'}z_1, w_2 + 2^{j'}z_2)\overline{\hat{\Phi}_{j';0,0}(w_1 + 2^{j'}z_1, w_2 + 2^{j'}z_2)}. \end{aligned}$$

For the second equality, one uses the poisson equality. Since $\hat{\theta}_\alpha$ and $\hat{\Phi}_{j';0,0}$ decrease fast at infinity, the series can be truncated. The scaling coefficient $C_{j';k_1,k_2}$ can then be recovered using a discrete Fourier transform. Finally, since the point values of $\Phi_{\alpha'}$ are known, it is also true for these of θ_α.

Chapter 2

Méthodes de domaine fictif

Le deuxième chapitre est consacré à la présentation des méthodes de domaine fictif avec multiplicateurs de Lagrange de bord pour la résolution numérique d'équations elliptiques. Premièrement, la méthode est présentée dans le cadre abstrait. Elle est appliqué à des équations elliptiques avec des conditions de Dirichlet sur la frontière. Les discrétisations de cette formulation utilisant la méthode de type Galerkin/ondelettes ou la méthode de Petrov-Galerkin/ondelettes sont présentés. Finalement nous mettons en evidence deux difficultés associées à ces méthodes:

Premièrement, le manque de regularité global de la solution limite le taux de convergence des solutions numériques. Deuxièmement l'utilisation explicite des opérateurs de trace conduit à des mauvais conditionnements numériques qui sont susceptible de déstabiliser les calculs ou de limiter les ordres de convergence. Nous introduisons alors les directions principales de notre travail qui reposent sur l'utilisation des bases d'ondelettes, le contrôle des conditions aux limites par des variables de contrôlde vivant sur ure courbe Γ à une distance non nulle de la frontière γ et la définition d'opérateurs de prolongement régulières.

Fictitious domain methods stand for a large class of methods that consist in solving a problem raised on a domain ω using a method that involves the merging of this domain ω into a larger and simpler domain Ω. When the initial problem is a partial differential equation for the unknown $\tilde{u}$ on ω, the fictitious domain method introduces a new partial differential equation on Ω such that its solution u(or one element of its solution, or one element when one parameter goes to zero) coincides with u when it is restricted to ω.

The different fictitious domain methods differs fundamentally in the vary boundary conditions are enforced and in the techniques used to extend the partial differential equation on $\Omega \setminus \omega$. In penalization methods additional terms are added in $\Omega \setminus \omega$. These terms are usually weighted with a coefficient $\frac{1}{\epsilon}$ in the way that when ϵ goes to zero the boundary condition are enforced[2],[45]. The introduction of Lagrange multipliers comes from optimization techniques under constraints. Minimization problem on a subset is transformed into a saddle point problem on a larger set. For PDE problems, Lagrange multipliers are supplementary unknowns that may live on ω, its boundary γ or $\Omega \setminus \omega$.

In the following, we focus on fictitious domain methods for a PDE raised on ω including Dirichlet boundary conditions. They lead to a new PDE raised on Ω which unknown are (u, λ), with λ being Lagrange multipliers on γ.

2.1 Méthodes de domaine fictif avec multiplicateurs de Lagrange surfaciques: formulation faible

Let $X(\omega)$, resp., $Y(\gamma)$ be Hilbert space on ω equipped with norm $\|\cdot\|_\omega$, resp., on γ with norm $\|\cdot\|_\gamma$. We make the following hypotheses:

Hypothesis 2.1. There exists a linear continuous mapping T_ω from $X(\omega)$ onto $Y(\gamma)$.

Introducing the subspace $X_g(\omega)$ of $X(\omega)$ defined as

$$X_g(\omega) = \{v \in X(\omega) \mid T_\omega(v) = g\},$$

the starting problem in an abstract form is:

$$\text{Find } \tilde{u} \in X_g(\omega) \text{ such that } \forall v \in X_0(\omega), a_\omega(\tilde{u}, v) = l_\omega(v), \tag{2.1}$$

where a_ω is a bilinear form continuous on $X(\omega)$ and coercive on $X_0(\omega)$, l_ω is a linear form continuous on $X(\omega)$. The abstract formulation for an elliptic equation on a domain ω transformed using a fictitious domain method with multipliers on γ (the boundary of ω) reads:

$$\begin{cases} \text{Find } (u, \lambda) \in X(\Omega) \times Y'(\gamma) \text{ such that} \\ a_\Omega(u, v) + (\lambda, T_\gamma(v)) = l_\Omega(v), \quad \forall v \in X(\Omega), \\ (\mu, T_\gamma(u)) = (\mu, g), \quad \forall \mu \in Y'(\gamma), \end{cases} \tag{2.2}$$

where $X(\Omega)$ is a Hilbert space on Ω equipped with norm $\| \cdot \|_\Omega$, $Y'(\gamma)$ is the dual space of $Y(\gamma)$, and its norm $\| \cdot \|_{Y'(\gamma)}$ defined by

$$\|\theta\|_{Y'(\gamma)} = \sup_{\mu \in Y(\gamma)} \frac{< \theta, \mu >}{\|\mu\|_{Y(\gamma)}},$$

a_Ω a continuous bilinear form on $X(\Omega)$, l_Ω a continuous linear form on $X(\Omega)$. In addition, we suppose that:

Hypothesis 2.2. There exists a continuous bilinear form a_Ω on $X(\Omega)$ which is coercive on $X(\Omega)$ and such that: there exists a bilinear continuous form $a_{\Omega\setminus\bar{\omega}}$ defined on $R_{\Omega\setminus\bar{\omega}}(X(\Omega))$ which satisfies

$$\forall (U, V) \in (X(\Omega))^2, a_\Omega(U, V) = a_\omega(u_1, v_1) + a_{\Omega\setminus\bar{\omega}}(u_2, v_2),$$

where $u_1 = R_\omega(U)$, $u_2 = R_{\Omega\setminus\bar{\omega}}(U)$, $v_1 = R_\omega(V)$, $v_2 = R_{\Omega\setminus\bar{\omega}}(V)$. Restriction operator R_ω, $R_{\Omega\setminus\bar{\omega}}$ are defined from $X(\Omega)$ to ω and $\Omega \setminus \bar{\omega}$, respectively.

In the same way, we suppose that there exists a continuous linear form l_Ω on $X(\Omega)$ and a linear continuous form $l_{\Omega\setminus\bar{\omega}}$ defined on $R_{\Omega\setminus\bar{\omega}}(X(\Omega))$ such that

$$\forall V \in X(\Omega), l_\Omega(V) = l_\omega(v_1) + l_{\Omega\setminus\bar{\omega}}(v_2).$$

Hypothesis 2.3. There exists a linear continuous mapping T_γ from $X(\Omega)$ onto $Y(\gamma)$ such that

$$\forall v \in X(\Omega), R_\omega(v) \in X_g(\omega) \Longleftrightarrow T_\gamma(v) = g.$$

The last one concerns the extension of $X_0(\omega)$ in $X(\Omega)$:

Hypothesis 2.4. Let u be an element of $X(\Omega)$ such that $T_\gamma(u) = 0$. Then, for any v in $X_0(\omega)$, the function V defined by

$$V = \begin{cases} v \text{ on } \omega, \\ R_{\Omega\setminus\bar{\omega}}(u) \text{ on } \Omega \setminus \bar{\omega}, \end{cases}$$

belongs to $X(\Omega)$.

Then, following [67] we have

Theorem 2.1. *Under hypothesis 2.1, 2.2, 2.3, 2.4, the problem (2.2) has a unique solution. Moreover $R_\omega(u)$ is the solution of problem (2.1). λ is the Lagrange multiplier.*

We illustrate the derivation of the fictitious domain formulation for the classical following Dirichlet problem. Let L be an elliptic operator, $f \in L^2(\omega)$, $g \in H^{\frac{1}{2}}(\gamma)$

$$\begin{cases} \text{find } u \in H^1(\omega) \text{ such that} \\ Lu = f \quad \text{on} \quad \omega, \\ u = g \quad \text{on} \quad \gamma, \end{cases} \tag{2.3}$$

First we note $X(\omega) = H^1(\omega)$, $Y(\gamma) = H^{\frac{1}{2}}(\gamma)$, T_ω is the trace operator from $H^1(\omega)$ to $H^{\frac{1}{2}}(\gamma)$.

Lemma 2.2. *Hypothesis 2.1 is satisfied, i.e. T_ω is linear continuous mapping from $H^1(\omega)$ onto $H^{\frac{1}{2}}(\gamma)$.*

We denote

$$X_0(\omega) = H_0^1(\omega) \text{ and } X_g(\omega) = \{v \in H^1(\omega) \mid T_\omega(v) = g\}.$$

The weak formulation of problem (2.3) reads

$$\begin{cases} \text{find } u \in X_g(\omega) \text{ such that} \\ a_\omega(u, v) = l_\omega(v), \quad \forall v \in X_0(\omega), \end{cases} \tag{2.4}$$

with

$$a_\omega(u,v) =< Lu, v >, \text{ and } l_\omega(v) =< f, v >, \forall u, v \in H^1(\omega).$$

Lemma 2.3. *Since a_ω is coercive on $H^1_0(\omega)$, then problem (2.4) has a unique solution, which is the weak solution of problem (2.3).*

Let $\tilde{f}$ be an extension of f in $L^2(\Omega)$, $X(\Omega) = H^1(\Omega)$. L is extended to function defined on Ω(still denoted as L),

$$a_\Omega(u,v) =< Lu, v >, \text{ and } l_\Omega(v) =< \tilde{f}, v >, \forall u, v \in H^1(\Omega).$$

Lemma 2.4. *Let a_Ω be coercive on $H^1(\Omega)$, then hypothesis 2.2 is satisfied.*

Then we define T_γ as

$$T_\gamma \begin{cases} H^1(\Omega) \to H^{\frac{1}{2}}(\gamma), \\ v \mapsto T_\omega(R_\omega(v)). \end{cases} \tag{2.5}$$

Lemma 2.5. *For $X(\Omega) = H^1(\Omega)$, $X_g(\omega) = \{v \in H^1(\omega) \mid T_\omega(v) = g\}$, $Y(\gamma) = H^{\frac{1}{2}}(\gamma)$, the hypothesis 2.2, 2.3 are satisfied.*

All these results lead to

Theorem 2.6. *Let a_Ω be coercive on $H^1(\Omega)$. Then, the problem*

$$\begin{cases} \text{find } (u,\lambda) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\gamma) \text{ such that} \\ a_\Omega(u,v) + (\lambda, T_\gamma(v)) = l_\Omega(v), \quad \forall v \in H^1(\Omega), \\ (\mu, T_\gamma(u)) = (\mu, g), \quad \forall \mu \in H^{-\frac{1}{2}}(\gamma), \end{cases} \tag{2.6}$$

has a unique solution. Moreover $R_\omega(u)$ is the solution of problem (2.3).

We often write (2.6) in matrix from. Indeed $(u,\lambda) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\gamma)$ satisfies

$$\begin{pmatrix} L & (T_\gamma)^* \\ T_\gamma & 0 \end{pmatrix} \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} \tilde{f} \\ g \end{pmatrix}. \tag{2.7}$$

where upper index $*$ denotes its adjoint operator. This linear system can also be derived from a *saddle point problem.* If one looks for the minimizer in $H^1(\Omega)$ of the functional

$$F(v) := \frac{1}{2} a_\Omega(v,v) - < \tilde{f}, v > \quad \forall v \in H^1(\Omega), \tag{2.8}$$

subject to the additional constraint $T_\gamma v = g$, then one can express the constraint in the equivalent form

$$(q, T_\gamma v) = (q, g), \quad \forall q \in H^{-1/2}(\gamma). \tag{2.9}$$

To solve this constrained minimization problem, one can append (2.9) to (2.8) using a Lagrange multiplier. Finally, the problem reads: find $\lambda \in H^{-1/2}(\gamma)$, $u \in H^1(\Omega)$, such that

$$\tilde{F}(u, \lambda) := \sup_{q \in H^{-1/2}(\gamma)} \inf_{v \in H^1(\Omega)} \tilde{F}(v, q), \tag{2.10}$$

where

$$\tilde{F}(v, q) = \frac{1}{2} a_\Omega(v, v) - < \tilde{f}, v > + (T_\gamma v, q) - (g, q).$$

Using standard variational arguments one concludes that this continuous saddle point problem is equivalent to (2.6). The existence and uniqueness of its solution is ensured by the following result in [12], [36]:

Theorem 2.7. *Let the linear operator L(L^* being its adjoint operator) be invertible on on $kerT_\gamma \subseteq H^1(\Omega)$, i.e., for some constant $\alpha_1 > 0$*

$$\inf_{v \in kerT_\gamma} \sup_{\omega \in kerT_\gamma} \frac{< Lv, \omega >_{H^{-1}(\Omega) \times H^1(\Omega)}}{\|v\|_{1,\Omega} \|\omega\|_{1,\Omega}} \geq \alpha_1, \quad \inf_{v \in kerT_\gamma} \sup_{\omega \in kerT_\gamma} \frac{< L^* v, \omega >_{H^{-1}(\Omega) \times H^1(\Omega)}}{\|v\|_{1,\Omega} \|\omega\|_{1,\Omega}} \geq \alpha_1, \tag{2.11}$$

and let for some constant $\beta_1 > 0$ the inf-sup condition

$$\sup_{v \in H^1(\Omega)} \frac{< T_\gamma v, q >_{H^{1/2}(\gamma) \times H^{-1/2}(\gamma)}}{\|v\|_{1,\Omega}} \geq \beta_1 \|q\|_{-1/2,\gamma}, \quad q \in H^{-1/2}(\gamma), \tag{2.12}$$

hold. Then for any $f \in H^{-1}(\Omega)$ and $g \in H^{1/2}(\gamma)$, there exists a unique solution $(u, \lambda) \in H^1(\Omega) \times H^{-1/2}(\gamma)$ to the problem (2.7). In other words,

$L := \begin{pmatrix} L & (T_\gamma)^* \\ T_\gamma & 0 \end{pmatrix}$ *is an isomorphism from $H^1(\Omega) \times H^{-1/2}(\gamma)$ to $H^{-1}(\Omega) \times H^{1/2}(\gamma)$,*

and one has the equivalence

$$\left\| \begin{pmatrix} v \\ q \end{pmatrix} \right\|_{H^1(\Omega) \times H^{-1/2}(\gamma)} \sim \left\| L \begin{pmatrix} v \\ q \end{pmatrix} \right\|_{H^{-1}(\Omega) \times H^{1/2}(\gamma)},$$

for any $(v, q) \in H^1(\Omega) \times H^{-1/2}(\gamma)$, where the constants contain α_1, β_1 and the continuity constants for L and T_γ.

Remark 2.1.1. The restriction to ω of u, $u\mid_\omega$ is the unique solution of problem (2.3)[37].

Using lemma 2.7, the proof of existence and uniqueness of problem (2.6) turns to the proof of conditions (2.11) and (2.12). For the latter, we first recall the following lemma [47]:

Lemma 2.8. *Let $\Omega \subseteq \square$ and smooth subsets $\Gamma \subset \partial\Omega$. If B stands for the trace operator on Γ, then for any $1/2 < s < 3/2$, there exists $c_{s,1}$ such that for any $f \in H^s(\square)$*

$$\|Bf\|_{H^{s-1/2}(\Gamma)} \leq c_{s,1}\|f\|_{H^s(\square)}.$$

The constant $c_{s,1}$ depends on $\square$ and Γ.

Conversely there exists for any $1/2 < s < 3/2$ a constant $c_{s,2}$ such that for any $h \in H^{s-1/2}(\Gamma)$, there exists $f \in H^s(\square)$ such that $Bf = h$ and

$$\|f\|_{H^s(\square)} \leq c_{s,2}\|h\|_{H^{s-1/2}(\Gamma)}.$$

The positive constant $c_{s,2}$ depends on $\square$ and Γ.

We can prove that the inf-sup condition (2.12) is satisfied for T_γ defined in (2.5): By the definition of the norm $\|\cdot\|_{H^{-1/2}(\Gamma)}$,

$$\begin{aligned} \|q\|_{H^{-1/2}(\gamma)} &= \sup_{p \in H^{1/2}(\gamma)} \frac{(p,q)_{H^{1/2}(\gamma)\times H^{-1/2}(\gamma)}}{\|p\|_{H^{1/2}(\gamma)}} \\ &= \sup_{\substack{v \in H^1(\Omega) \\ T_\gamma v \in H^{1/2}(\gamma)}} \frac{(T_\gamma v, q)}{\|T_\gamma v\|_{H^{1/2}(\gamma)}} \\ &\leq \sup_{v \in H^1(\Omega)} \frac{c_{s,2}(T_\gamma v, q)}{\|v\|_{H^1(\Omega)}}. \end{aligned}$$

Therefor inf-sup condition (2.12) holds true with $\beta_1 = \frac{1}{c_{s,2}}$. Finally, as soon as the continuous ellipticity condition (2.11) holds, then problem (2.6) has an unique solution.

Remark 2.1.2. If L is symmetric, and coercive on $\ker T_\gamma$, then (2.11) holds. For the Dirichlet problem 2.6, the Lagrange multiplier on $\gamma(\lambda)$ is the jump of the conormal derivative on γ. This result is easily obtained checking that the Lagrange multiplier can be derived from the weak formulation (2.6) using Green's formula. It follows that when the Lagrange multiplier is non zero, $u \in H^s(\Omega)$, $s \leq 3/2 - \epsilon$, $\epsilon > 0$. We note from [39], [38] that the Lagrange multiplier has a physical meaning when the right hand side f is extended by zero. It is the stress vector on the boundary.

2.2 Approximation de type Galerkin

2.2.1 Résultats généraux sur la méthode de Galerkin

If we use a Galerkin method to approximate problem (2.2) and if for $h > 0$, X_h (resp Y'_h) are finite dimensional subspaces of $X(\Omega)$ (resp $Y'(\gamma)$), then we look for a solution to the following problem,

$$\begin{cases} \text{find } (u_h, \lambda_h) \in X_h \times Y'_h \text{ such that} \\ a_\Omega(u_h, v_h) + (\lambda_h, T_\gamma(v_h)) = l_\Omega(v_h), \quad v_h \in X_h, \\ (q_{h'}, T_\gamma(u_h)) =< q_{h'}, g >, \quad \forall q_{h'} \in Y'_h. \end{cases} \tag{2.13}$$

It reads: Find $(u_h, \lambda_h) \in X_h \times Y'_h$ such that

$$\begin{pmatrix} L_h & (T_\gamma)_h^T \\ (T_\gamma)_h & 0 \end{pmatrix} \begin{pmatrix} u_h \\ \lambda_h \end{pmatrix} = \begin{pmatrix} l_\Omega \\ g \end{pmatrix}. \tag{2.14}$$

where the upper index T denotes its transpose. For the proof of existence and uniqueness of discrete problem (2.13), we need two important conditions, i.e. the discrete ellipticity condition and *Ladyšenskaja-Babuška-Brezzi (LBB) condition*. In the following, we will introduce them in details. First, the discrete ellipticity condition reads as follows: the discrete linear operator L_h is required to be invertible on $ker(T_\gamma)_h$ with some constant $\alpha_1 > 0$ independent of h,

$$\inf_{v \in ker(T_\gamma)_h} \sup_{\omega \in ker(T_\gamma)_h} \frac{< L_h v, \omega >_{X' \times X}}{\|v\|_X \|\omega\|_X} \geq \alpha_1, \quad \inf_{v \in ker(T_\gamma)_h} \sup_{\omega \in ker(T_\gamma)_h} \frac{< L_h^T v, \omega >_{X' \times X}}{\|v\|_X \|\omega\|_X} \geq \alpha_1, \tag{2.15}$$

with

$$ker(T_\gamma)_h = \{v_h \in X_h : (q_{h'}, T_\gamma(v_h)) = 0, \forall q_{h'} \in Y'_h\}.$$

Secondly, the LBB condition that describes the compatibility of the finite dimension spaces X_h and Y'_h reads: There exists a constant β independent of h such that

$$\sup_{v_h \in X_h, v_h \neq 0} \frac{(q_{h'}, T_\gamma(v_h))}{\|v_h\|_X \|q_{h'}\|_{Y'}} \geq \beta, \quad \forall q_{h'} \in Y'_h. \tag{2.16}$$

Theorem 2.9. *[47] If the discrete ellipticity condition (2.15) and the LBB condition (2.16) hold, then the discrete problem (2.13) has a unique solution* (u_h, λ_h).

Remark 2.2.1. As in the continuous case, if L_h is symmetric, the discrete ellipticity condition can be replaced by $a_\Omega(\cdot,\cdot)$ is coercive on $Ker(T_\gamma)_h$.

Remark 2.2.2. The main result in [25] showed that the LBB condition is satisfied whenever the discretization step length on the boundary, $h_\gamma \sim 2^{-j'}$, is somewhat bigger than the one on the domain, $h_\Omega \sim 2^{-j}$. In other words, $j - j' > B$, where the constant B stems from the trace theorem, norm equivalence for the Lagrange space on the boundary, Jackson and Berstein estimates.

From the continuity of a_Ω and T_γ, it follows

$$|a_\Omega(u,v)| \leq c_1 \|u\|_X \|v\|_X, \quad |(\lambda, T_\gamma(u))| \leq c_2 \|u\|_X \|\lambda\|_{Y'},$$

for all $u, v \in X$, $\lambda \in Y'$. Now it comes to the error estimate [63],

Theorem 2.10. *Let the conditions in theorems 2.7, theorem 2.9 be satisfied. If* (u, λ) *and* (u_h, λ_h) *denote the solutions of problems (2.2) and (2.13), then we have*

$$\|u - u_h\|_X \leq (1 + \frac{c_1}{\alpha}) \inf_{\tilde{u} \in X_h^g} \|u - \tilde{u}\|_X + \frac{c_2}{\alpha} \inf_{\tilde{\lambda} \in Y'_h} \|\lambda - \tilde{\lambda}\|_{Y'}.$$

$$\|\lambda - \lambda_h\|_{Y'} \leq \frac{c_1}{\beta}(1 + \frac{c_1}{\alpha}) \inf_{\tilde{u} \in X_h^g} \|u - \tilde{u}\|_X + (1 + \frac{c_2}{\beta} + \frac{c_1 c_2}{\alpha\beta}) \inf_{\tilde{\lambda} \in Y'_h} \|\lambda - \tilde{\lambda}\|_{Y'},$$

with $X_h^g = \{v_h \in X_h, (q_{h'}, T_\gamma(v_h)) = (q_{h'}, g), \forall q_{h'} \in Y'_h\}$.

2.2.2 Ondelettes/méthode de Galerkin

In wavelet discretization setting, the spaces X_h and Y'_h are constructed as follows:

— X_h =span$\{\phi_\alpha^\Omega, \alpha \in K_j\}$ where ϕ_α^Ω are the orthogonal scaling functions of an m-regular multi-resolution analysis of X, $m > 0$.

— Y_h' =span$\{\phi_{\alpha'}^\gamma, \alpha' \in K_{j'}'\}$ where $\phi_{\alpha'}^\gamma$ are the orthogonal scaling functions of an m'-regular multi-resolution analysis of Y', $m' > 0$.

with subset of indices

$K_j = \{\alpha = (j, k_1, k_2), k_i = 0, \cdots, 2^j - 1, i = 1, 2\}$ with $\mid \alpha \mid = j$.

$K_{j'}' = \{\alpha' = (j', l_1'), l_1' = 0, \cdots, 2^{j'} - 1\}$ with $\mid \alpha' \mid = j'$.

The LBB condition is described as follows[25]:

Theorem 2.11. *If X_h and Y_h' are constructed as above and $Y_h' \in H^{s'-1/2}(\gamma)$ with $s' > 1$, then the LBB condition holds if $j - j' > B$, where B is a constant that depends on the problem.*

2.2.3 Résultats généraux sur la méthode de Petrov-Galerkin

In a Petrov-Galerkin method, two spaces $X_h \subset X$ and $\tilde{X}_h \subset X$ are introduced as well as $Y_h' \subset Y'$. It reads

$$\begin{cases} \text{find } (u_h, \lambda_h) \in X_h \times Y_h' \text{ such that} \\ a_\Omega(u_h, v_h) + (\lambda_h, T_\gamma(v_h)) = l_\Omega(v_h), \quad v_h \in \tilde{X}_h, \\ (q_{h'}, T_\gamma(u_h)) =< q_{h'}, g >, \quad \forall q_{h'} \in Y_h'. \end{cases} \tag{2.17}$$

The following theorem [6] gives the conditions for the existence and uniqueness of the solution of problem (2.17).

Theorem 2.12. *Let $X_h \subset X$, $\tilde{X}_h \subset X$ and $Y_h' \subset Y'$ be three approximation spaces. Assume that there exist two positive constants K and ζ such that the bilinear form $a_\Omega(\cdot, \cdot)$ satisfies*

$$|a_\Omega(u_h, v_h)| \leq K \|u_h\|_X \|v_h\|_X, \quad \forall u_h \in X_h, \forall v_h \in \tilde{X}_h,$$

$$\sup_{v_h \in \tilde{X}_h, v_h \neq 0} \frac{a_\Omega(u_h, v_h)}{\|v_h\|_X} \geq \zeta \|u_h\|_X, \quad \forall u_h \in X_h,$$

$$\sup_{u_h \in X_h} a_\Omega(u_h, v_h) > 0, \quad \forall v_h \in \tilde{X}_h, v_h \neq 0.$$

Moreover, assume that there exist two positive constants β_1 and β_2 such that

$$\sup_{v_h \in \tilde{X}_h, v_h \neq 0} \frac{(q_{h'}, T_\gamma(v_h))}{\|v_h\|_X} \geq \beta_1 \|q_{h'}\|_{Y'}, \quad \forall q_{h'} \in Y'_h, \tag{2.18}$$

$$\sup_{u_h \in X_h, u_h \neq 0} \frac{(q_{h'}, T_\gamma(u_h))}{\|u_h\|_X} \geq \beta_2 \|q_{h'}\|_{Y'}, \quad \forall q_{h'} \in Y'_h.$$

Then, there exists a unique solution (u_h, λ_h) of problem (2.17).

2.2.4 Ondelettes/méthode de Petrov-Galerkin

$\tilde{X}_h$ is an approximation space of X and equals to span$\{\theta^e_{j,k}, k \in \mathbb{Z}^2, e \in E = \{\{0,1\}^2 \setminus (0, \cdots, 0)\}$ where $\theta^\Omega_\alpha = (L^*)^{-1}\Psi^e_{j,k}$.

A sufficient condition to satisfy the LBB condition is given in [6]:

Theorem 2.13. *If X_h, $\tilde{X}_h$ and Y'_h are constructed as above and $Q_h \in H^{s'-1/2}(\gamma)$ with $s' > 2$, then the LBB condition of problem (2.17) holds if $j - j' > B_1$, $j(s' - 2) - j'(s' - 1) > B_2$, where B_1 and B_2 are constants that depend on the problem.*

2.3 Limitations de ces méthodes

2.3.1 Régularité de u

Extension of f from ω to Ω

The extension of f from ω to Ω that is required to apply a fictitious domain method is critical since the smoothness of the extension $\tilde{f}$ influences the regularity of the solution on Ω and therefore of its restriction to ω.

Extension by a constant function(usually zero) is not a good solution since discontinuity along γ generate a poor quality approximation. It is generally difficult to combine smoothness of $\tilde{f}$ with the exact restriction condition $\tilde{f}|_\omega = f$. Using Fourier expansion for instance [30], the restriction condition is verified only on grid points while the smoothness of $\tilde{f}$ is optimized through a minimization of $\|\Delta^p \tilde{f}\|_{L^2(\Omega)}$.

Lagrange multipliers

Since the Lagrange multipliers stand for the jump in conormal direction at γ, when $\lambda \neq 0$ one gets a lack of regularity in the vicinity of γ. This phenomenon affects the regularity of u on the whole domain Ω. This is a real drawback of such fictitious domain method since in the original problem, the regularity of u on ω only depends on f. The convergence analysis between u and u_h on domain Ω shows that the decrease rate of the error is constrained by the largest s such that $u \in H^s(\Omega)$. Since $s \leq 3/2 - \epsilon$, the convergence rate is low.

Interior estimates

Interior estimates deal with the approximation error $e = u - u_h$ on a subdomain. J.A. Nische established in [56] a general setting for interior estimates. Let h be a parameter, k and r are given integer, $S^h_{k,r}(\Omega)$ denotes a one-parameter family of finite dimensional subspace of $H^k(\Omega)$. The following approximation assumption (2.19)-(2.11) are shared by many finite dimensional subspaces used in practice to approximate the solutions of partial differential equations. For example, $S^h_{k,r}(\Omega)$ is the restriction to Ω of Hermite splines defined on a uniform mesh with sides of length h. Here $k = m$, $r = 2m$, $m = 1, 2, \cdots$. Let $u \in H^l(\Omega)$, $u_h \in S^h_{k,r}(\Omega)$, if the error $e = u - u_h$ satisfies an interior equation:

$$a_\Omega(e, v_h) = 0, \quad \forall v_h \in \mathring{S}^h_{k,r}(\Omega),$$

where

$$\mathring{S}^h_{k,r}(\Omega) = \{v_h \in S^h_{k,r}(\Omega), \mathrm{supp} v_h \subset \Omega\}.$$

Under several approximation assumption concerning $S^h_{k,r}(\Omega)$: if $\Omega_0 \subset \Omega_1 \subset \Omega$

A.1. For each $u \in H^l(\Omega_1)$, there exists an $\eta \in S^h_{k,r}(\Omega)$ such that for any $0 \leq s \leq k$, $s \leq l \leq r$

$$\|u - \eta\|_{s,\Omega_1} \leq ch^{l-s}\|u\|_{l,\Omega_1}, \tag{2.19}$$

A.2. Let $w \in C^\infty_0(\Omega_0)$ and $u_h \in S^h_{k,r}(\Omega)$, then there exists an $\eta \in \mathring{S}^h_{k,r}(\Omega_1)$ such that

$$\|wu_h - \eta\|_{1,\Omega_1} \leq ch\|u_h\|_{1,\Omega_1}, \tag{2.20}$$

where $c = c(\Omega_0, \Omega_1, w)$.

A.3. For each $h \in (0, h_0]$, there exists a set Ω_h, $\Omega_0 \subset \Omega_h \subset \Omega_1$ such that, if $0 \leq v \leq s \leq k$, then for all $f \in S^h_{k,r}(\Omega)$

$$\|f\|_{s,\Omega_h} \leq ch^{v-s}\|f\|_{v,\Omega_h}. \tag{2.21}$$

Moreover, $p \geq 0$ is an arbitrary but fixed integer. Then there exists h_1 such that for all $h \leq h_1$, if $s = 0, 1$, $1 \leq l \leq r$,

$$\|e\|_{s,\Omega_0} \leq c(h^{l-s}\|u\|_{l,\Omega_1} + \|e\|_{-p,\Omega_1}),$$

if $2 \leq s \leq l \leq r$, $s \leq k < r$, then

$$\|e\|_{s,\Omega_0} \leq c(h^{l-s}\|u\|_{l,\Omega_1} + h^{1-s}\|e\|_{-p,\Omega_1}).$$

Taking p large enough, the second term on the right of estimates becomes neglectable. These estimates stand in our problem on any interior subdomain of ω but not on ω itself.

S. Bertoluzza has generalized these results to the wavelet Galerkin method, where the space $S^h_{k,r}(\Omega)$ is spanned by compact support wavelet basis. Indeed as the wavelet used is compactly supported, we easily derive the local Jackson and Berstein inequalities, which in turn are the assumptions (2.19) and (2.21). From the norm equivalence and compactly supported wavelet basis, it is proven that the assumption (2.20) also holds. So the interior error estimates can be applied to wavelet Galerkin method.

Directions of our work

The previously mentioned limitations open the following direction of our work.

(1) Smooth extension

We will propose an extension procedure based on the construction of multiresolution analysis on the interval. This procedure will be defined and analyzed in 1D and generalized to the bivariate situation. It is the subject of chapter 4.

(2) Control boundary

As it has been recalled the introduction of Lagrange multipliers on γ affects the regularity of u in the vicinity of γ. We propose to use a control boundary $\Gamma \neq \gamma$, laying at a positive distance from γ in the outer normal direction. We will analyze the corresponding method and provide numerical results in the following chapter 3.

(3) Interior estimates for Petrov-Galerkin method

We will derive interior estimates associated to wavelet Petrov-Galerkin methods and non compact wavelets in chapter 3.

2.3.2 Conditionnement

The iterative matrix involved in solving the discretization problem is usually ill-conditioned. For example, using the Petrov-Galerkin wavelet method [5], it has been proved that $\exists K < \infty$, $\mathrm{cond}_2((DC)^t(DC)) \leq K4^{j'+2j}$, with j and j' the discretization level on Ω and γ, respectively. The numerical results show that the condition number conforms well with respect to j', namely $4^{j'}$. However as mentioned in [6],[47], this estimate is very pessimistic with respect to j. We will revisit this estimate and the associated preconditioning procedure in chapter 3.

Chapter 3

Définition et analyse d'une méthode de domaine fictif lisse pour un problème elliptique

Ce chapitre introduit une méthode de domaine fictif lisse pour des équations elliptiques. Cette nouvelle méthode utilise une courbe de contrôle Γ située à distance positive de la frontière γ du domaine original. L'analyse de l'existence de la solution de la formulation faible suit les travaux des J. Haslinger et al[40]. Cette analyse contraint la régularité de la condition aux limites sur γ à un sous-espace strict mais dense de $H^{1/2}(\gamma)$. Cette contrainte correspond à un résultat de contrôlabilité approchée de la trace d'un problème elliptique sur une frontiére intérieure. Ensuite, nous discrétisons le problème en utilisant une analyse multirésolution biorthognale sur $H^1_{\mathcal{P}}(\Omega)$ et des analyses multirésolution orthogonales sur $H^{-\frac{1}{2}}(\gamma)$ et $H^{-\frac{1}{2}}(\Gamma)$. Nous appelons cette méthode, méthode Petrov-Galerkin/ondelettes. L'inversibilité de la matrice dans le système de point de selle assure de l'existence de probléme discret. Ensuite, nous prouvons la condition LBB , qui décrit la compatibilité entre les différents sous-espaces et qui joue un rôle important. Nous obtenous également deux types d'estimations d'erreur, l'une globale c'est à dire sur l'espace Ω tout entier, l'autre locale c'est à dire valable sur tout ouvert ω_0 avec $\bar{\omega}_0 \subset \Xi$.

L'analyse du conditionnement du problème linéaire montre qu'il est indépendant de j(l'échelle sur Ω) en dimension 1 mais qu'il reste contrôlé par $2^{2j+j+j'}$(l'échelle j' sur Γ, l'échelle j'' sur γ) en dimension 2. Un préconditionnement diagonal est cependant disponible. Différents exemples numériques illustrent les différentes estimations globale et intérieure et permettent la comparaison entre la méthode de domaine fictif classique et la méthode domaine fictif lisse.

In [40], J.Haslinger, T.Kozubek, R.Kucera and G.Peichl investigated a new formulation of fictitious doamin method in introducing a new control variable, instead of a Lagrange multiplier, defined on a control boundary Γ located outside of $\bar{\omega}$ to enforce the boundary condition on γ. Using this formulation, the Dirichlet boundary value problem is transformed into the following problem: find $(u, \lambda) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ such that

$$\begin{cases} \text{Find } (u, \lambda) \in H^1(\Omega) \times H^{-1/2}(\Gamma) \text{ such that,} \\ a_\Omega(u, v) + b_1(v, \lambda) =< \tilde{f}, v > \quad v \in H^1(\Omega), \\ b_2(u, \mu) =< g, \mu > \quad \forall \mu \in H^{-1/2}(\gamma), \end{cases}$$

where the bilinear form b_1 and b_2 are distinct. This kind of problem can be considered as a generalization of saddle point problems and has been considered in [55] and [8]. The discrete problem resulting from discretization is non symmetric. In the following, we will use a Petrov-Galerkin method to discretize this problem. Since the derivative jump λ lays on Γ at a positive distance from γ, this new method is expected to produce a smooth solution on Ξ the domain bounded by Γ with $\bar{\omega} \subset \Xi$. We therefore expect to have a better convergence rate using interior error estimates on Ξ even if the global estimate remains poor since the regularity of u on Ω is still $H^{3/2-\epsilon}(\Omega)$, for any $\epsilon > 0$.

3.1 Introduction

Starting from the boundary-value problem introduced in section 1.7, we suppose that L reads:

$$Lu := -\sum_{i,j=1}^{n} (a_{ij}(x) u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i(x) u_{x_i} + c(x) u,$$

Therefore, the adjoint operator L^* of L reads:

$$L^* v := -\sum_{i,j=1}^{n} (a_{ij}(x) v_{x_j})_{x_i} - \sum_{i=1}^{n} b_i(x) v_{x_i} + (c(x) - \sum_{i=1}^{n} (b_i(x))_{x_i}) v.$$

In our following discussion, we assume that there exists constants $c_{a,1}$ and $c_{a,2}$ such that

$$c_{a,1}\|v\|_{H^s(\Omega)} \leq \|L^*v\|_{H^{s-2}(\Omega)} \leq c_{a,2}\|v\|_{H^s(\Omega)}, s \geq 2, c_{a,i} > 0, i = 1, 2. \tag{3.1}$$

For example, if $L = I - \nu\triangle$ with $\nu > 0$, one gets $c_{a,1} = \min(1, 4\pi^2\nu)$ and $c_{a,2} = \max(1, 4\pi^2\nu)$.

The weak formulation of problem (1.16) reads as follows:

$$\begin{cases} \text{Find } u \in H^1(\omega), \text{ such that } u = g \text{ on } \gamma \text{ and} \\ A(u, v) = (f, v) \quad \forall v \in H_0^1(\omega). \end{cases} \tag{3.2}$$

When L is a second order elliptic operator, A is defined like in (1.18). In addition, the original boundary γ is assumed to be smooth enough. Instead of (3.2), we consider the extended problem using the smooth fictitious domain approach. Let Γ be the boundary of a new domain Ξ with $\overline{\omega} \subset \Xi$ and $\overline{\Xi} \subset \Omega$(see figure 3.1). Then the new problem reads:

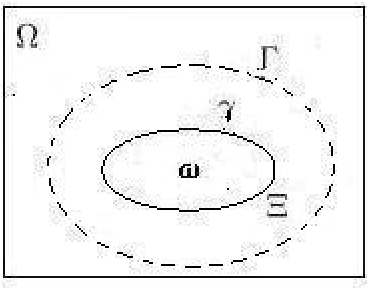


Figure 3.1: Geometry

$$\begin{cases} \text{Find } u \in H^1_{\mathcal{P}}(\Omega), \lambda \in H^{-\frac{1}{2}}(\Gamma), \text{ such that} \\ a_\Omega(u, v) + (T_\Gamma v, \lambda)_\Gamma = (\tilde{f}, v), \quad \forall v \in H^1_{\mathcal{P}}(\Omega), \\ (T_\gamma u, \mu)_\gamma = (g, \mu)_\gamma, \qquad \forall \mu \in H^{-\frac{1}{2}}(\gamma), \end{cases} \tag{3.3}$$

where the following notations are introduced

$$a_\Omega(u, v) := \int_\Omega \sum_{i,j=1}^n a_{ij}(x) u_{x_i} v_{x_j} + \sum_{i=1}^n b_i(x) u_{x_i} v + c(x) uv.$$

$$T_\Gamma \begin{cases} H^1_{\mathcal{P}}(\Omega) \rightarrow H^{\frac{1}{2}}(\Gamma) \\ u \mapsto T_\Xi(R_\Xi(u)), \end{cases} \tag{3.4}$$

$$T_\gamma \begin{cases} H^1_{\mathcal{P}}(\Omega) \rightarrow H^{\frac{1}{2}}(\gamma) \\ u \mapsto T_\omega(R_\omega(u)). \end{cases}$$

$H^1_{\mathcal{P}}(\Omega)$ stands for the periodic Sobolev space of order one. R_Ξ(resp.R_ω) is restriction operator from $H^1_{\mathcal{P}}(\Omega)$ to Ξ(resp.ω) while T_Ξ(resp.T_ω) is trace operator from $H^1(\Xi)$ (resp.$H^1(\omega)$) to $H^{1/2}(\Gamma)$(resp.$H^{1/2}(\gamma)$). $\tilde{f}$ represents an extension of f from ω to Ω, $\tilde{f} \in L^2(\Omega)$.

Suppose that (3.3) has a solution (u, λ). Then it is easy to see that (u, λ) satisfies the following equations and boundary conditions:

$$\begin{cases} Lu = \tilde{f} \quad \text{in} \quad \Xi \cup (\Omega \backslash \overline{\Xi}), \\ u = g \quad \text{on} \quad \gamma, \\ \text{periodic boundary condition on} \quad \partial\Omega, \\ \lambda = -[\dfrac{\partial u}{\partial n_L}]_\Gamma \quad \text{on} \quad \Gamma, \end{cases} \tag{3.5}$$

where n is the outward unit normal to Γ, $\frac{\partial u}{\partial n_L} := \sum_{i,j=1}^n a_{i,j}(x) u_{x_i} \vec{n}_j$ is the conormal derivative of u. It is obvious that $u|_\omega$ is the solution of original problem.

Remark 3.1.1. Since $Lu = \tilde{f}$ in Ξ, then if $\tilde{f}|_\Xi \in H^s(\Xi)$, $s \geq 0$, from the interior regularity theorem [32], we know that $u|_\Xi \in H^{s+2}(\Xi)$. This means that the solution u is smoothly extended from ω to Ξ. The regularity of u in Ξ depends on that of $\tilde{f}$. A new wavelet method of smooth extension of $\tilde{f}$ will be introduced in chapter 4.

3.2 Analyse continue

In the following, we assume that the bilinear form $a_\Omega(\cdot, \cdot)$ is continuous and coercive on $H^1_{\mathcal{P}}(\Omega)$. We use the work of J.Haslinger et al. in [40]. Since the first equation of (3.3) has a unique solution $u_\lambda \in H^1_{\mathcal{P}}(\Omega)$ for any $\lambda \in H^{-1/2}(\Gamma)$, a linear mapping Φ

is defined by:

$$\Phi \begin{cases} H^{-1/2}(\Gamma) \mapsto H^{1/2}(\gamma) \\ \lambda \mapsto T_\gamma(u_\lambda) \end{cases} \tag{3.6}$$

where T_γ stands for the trace on γ. The following lemma, proved in [40], describes the approximation controllability result.

Lemma 3.1. *The range $\Phi(H^{-1/2}(\Gamma))$ is dense in $H^{1/2}(\gamma)$.*

When $g \in \Phi(H^{-1/2}(\Gamma))$, the problem (3.3) has a solution. And $u|_\omega$ is the unique solution of the original problem. For $g \notin \Phi(H^{-1/2}(\Gamma))$, then for $\forall \epsilon > 0$, one can find $\tilde{g} \in \Phi(H^{-1/2}(\Gamma))$ such that

$$\|g - \tilde{g}\|_{H^{1/2}(\gamma)} \leq \epsilon.$$

Let $\hat{u}$ be a solution of (3.3) with $g = \tilde{g}$. Using the stability of an elliptic problem, the error of $u - \hat{u}$ is controlled by

$$\|u - \hat{u}\|_{H^1(\omega)} \leq c\|g - \tilde{g}\|_{H^{1/2}(\gamma)} \leq c\epsilon$$

where the c is the constant dependent on the continuity and coerciveness coefficient of bilinear form. From the inequality above, we see $\hat{u}|_\omega$ is a good approximation of the original problem.

3.3 Analyse discrete

Following [6],[33], we choose a Petrov-Galerkin and Wavelet-Vaguelette method to discretize (3.3). Finite dimension subspaces are introduced: $U_h^\Omega \subset H^1_{\mathcal{P}}(\Omega)$, $V_h^\Omega \subset H^1_{\mathcal{P}}(\Omega)$, $Q_{h'}^\Gamma \subset H^{-\frac{1}{2}}(\Gamma)$, $Q_{h''}^\gamma \subset H^{-\frac{1}{2}}(\gamma)$. Then the discretization problem reads:

$$\begin{cases} \text{Find } \; u_h \in U_h^\Omega, \lambda_{h'} \in Q_{h'}^\Gamma, \text{such that} \\ a_\Omega(u_h, v_h) + (T_\Gamma v_h, \lambda_{h'})_\Gamma = (\tilde{f}, v_h) \qquad \forall v_h \in V_h^\Omega, \\ (T_\gamma u_h, \mu_{h''})_\gamma = (g, \mu_{h''})_\gamma \qquad \forall \mu_{h''} \in Q_{h''}^\gamma. \end{cases} \tag{3.7}$$

If We note, for every $j \in \mathbf{N}$,

$K_j = \{\alpha = (j, k_1, \cdots, k_n), k_i = 0, \cdots, 2^j - 1, i = 1, \cdots, n\}$ with $\mid \alpha \mid= j$,

$K'_{j'} = \{\alpha' = (j', l'_1, \cdots, l'_{n-1}), l'_i = 0, \cdots, 2^{j'} - 1, i = 1, \cdots, n-1\}$ with $| \alpha' |= j'$,
$K''_{j''} = \{\alpha'' = (j'', s''_1, \cdots, s''_{n-1}), s''_i = 0, \cdots, 2^{j''} - 1, i = 1, \cdots, n-1\}$ with $| \alpha'' |= j''$,
then the precise definitions of these subspaces are given as follows (see also [13],[6]).

Definition 3.2. — U_h^Ω =span$\{\phi_\alpha^\Omega, \alpha \in K_j\}$ where ϕ_α^Ω are the orthogonal scaling functions of an m-regular, n-order multi-resolution analysis of $H^1_{\mathcal{P}}(\Omega)$, $n > 0$.

—V_h^Ω =span$\{\theta_\alpha^\Omega, \alpha \in K_j\}$ with $\theta_\alpha^\Omega = (L^*)^{-1}\phi_\alpha^\Omega$.

— $Q_{h'}^\Gamma$ =span$\{\phi_{\alpha'}^\Gamma, \alpha' \in K'_{j'}\}$ where $\phi_{\alpha'}^\Gamma$ are the orthogonal scaling functions of an m'-regular, n'-order multi-resolution analysis of $H^{-\frac{1}{2}}(\Gamma)$, $n' > 0$.

— $Q_{h''}^\gamma$ =span$\{\phi_{\alpha''}^\gamma, \alpha'' \in K''_{j''}\}$ where $\phi_{\alpha''}^\gamma$ are the orthogonal scaling functions of an m"-regular, n"-order multi-resolution analysis of $H^{-\frac{1}{2}}(\gamma)$,$n'' > 0$.

Remark 3.3.1. If $\tilde{\theta}_\alpha^\Omega = L\phi_\alpha^\Omega$, with θ_α^Ω, then the couple $(\theta_{\alpha_1}^\Omega, \tilde{\theta}_{\alpha_2}^\Omega)$ fulfills the biorthogonality relation

$$(\theta_{\alpha_1}^\Omega, \tilde{\theta}_{\alpha_2}^\Omega) = \delta_{k_1,k'_1} \cdots \delta_{k_n,k'_n},$$

where $\alpha_1 = (j, k_1, \cdots, k_n)$ and $\alpha_2 = (j, k'_1, \cdots, k'_n)$. Then one can define the biorthogonal projector $P_{V_h^\Omega}$,

$$P_{V_h^\Omega} \begin{cases} H^1_{\mathcal{P}}(\Omega) \to V_h^\Omega \\ u \mapsto \sum_\alpha (u, \tilde{\theta}_\alpha^\Omega)\theta_\alpha^\Omega, \end{cases}$$

and an orthogonal projector $P_{U_h^\Omega}$,

$$P_{U_h^\Omega} \begin{cases} H^1_{\mathcal{P}}(\Omega) \to U_h^\Omega \\ u \mapsto \sum_\alpha (u, \phi_\alpha^\Omega)\phi_\alpha^\Omega. \end{cases}$$

The matrix form of the discrete problem reads:

$$\begin{cases} \text{Find } U_h \in I\!R^{d_h}, \Lambda_{h'} \in I\!R^{d_{h'}}, \text{ such that} \\ \begin{pmatrix} I & C \\ D & 0 \end{pmatrix} \begin{pmatrix} U_h \\ \Lambda_{h'} \end{pmatrix} = \begin{pmatrix} F_h \\ G_h \end{pmatrix} \end{cases} \tag{3.8}$$

when in 1D $d_h = 2^j$, $d_{h'} = 2$ while in 2D $d_h = 4^j$, $d_{h'} = 2^{j'}$. For the detail expression of the matrices involved see in the section 3.5. We use some notation to denote the null-space and the range-space of a matrix $M(d_{h'} \times d_h)$ on R^{d_h}

$$\ker(M) = \{v \in R^{d_h}; Mv = 0\},$$

$$\mathcal{R}(M) = \{u \in R^{d_{h'}}; u = Mv, v \in R^{d_h}\}.$$

In addition, let $\mathcal{A}$ be the matrix of problem (3.8),

$$\mathcal{A} = \begin{pmatrix} I & C \\ D & 0 \end{pmatrix}$$

Applying the theorem 3.1 in [40] which gives the necessary and sufficient condition for the invertibility of matrix $\mathcal{A}$. It states

Theorem 3.3. *$\mathcal{A}$ is invertible iff C has full column rank and the following condition hold*

$$\mathcal{R}(C) \cap kerD = \{0\}. \tag{3.9}$$

Remark 3.3.2. It is noted that the invertibility of $\mathcal{A}$ implies: C is of full column rank, D of full row rank and $j' = j''$. The we recall some properties about matrix rank. Let matrix A be of size $n \times m$, then rank$(A) \leq \min\{n, m\}$. So we get the necessary condition on the discretization scale on Ω, Γ and γ. They satisfy $j' \leq 2j$ and $j'' \leq 2j$.

Then we recall some important properties concerning wavelets (see also [19]),

Lemma 3.4. *For U_h^Ω, V_h^Ω, $Q_{h'}^\Gamma$ and $Q_{h''}^\gamma$ constructed as previously, the following Jackson estimates hold:*

$$\forall \lambda \in H^{s'}(\Gamma), 0 < s' \leq n', \inf_{\lambda_{h'} \in Q_{h'}^\Gamma} \|\lambda - \lambda_{h'}\|_{L^2(\Gamma)} \leq C_{J,\phi^\Gamma} 2^{-j's'} \|\lambda\|_{H^{s'}(\Gamma)},$$

$$\forall \mu \in H^{s''}(\gamma), 0 < s'' \leq n'', \inf_{\mu_{h''} \in Q_{h''}^\gamma} \|\mu - \mu_{h''}\|_{L^2(\gamma)} \leq C_{J,\phi^\gamma} 2^{-j''s''} \|\mu\|_{H^{s''}(\gamma)},$$

$$\forall u \in H^s(\Omega), 0 < s \leq n, \inf_{u_h \in U_h^\Omega} \|u - u_h\|_{L^2(\Omega)} \leq C_{J,\phi^\Omega} 2^{-js} \|u\|_{H^s(\Omega)},$$

$$\forall v \in H^s(\Omega), 2 < s \leq n+2, \inf_{v_h \in V_h^\Omega} \|v - v_h\|_{H^2(\Omega)} \leq C_{J,\theta^\Omega} 2^{-j(s-2)} \|v\|_{H^s(\Omega)}, \tag{3.10}$$

where the order of multiresolution analysis for $Q_{h'}^\Gamma$, $Q_{h''}^\gamma$, U_h^Ω is n', n'', n respectively.

The first three estimations are well known, we only need to verify the last one.

Proof.

$$\begin{aligned}
\|v - P_{V_h^\Omega} v\|_{H^2(\Omega)} &= \|v - \sum_\alpha (v, L\phi_\alpha^\Omega)(L^*)^{-1}\phi_\alpha^\Omega\|_{H^2(\Omega)} \\
&= \|(L^*)^{-1}(L^*v - \sum_\alpha (L^*v, \phi_\alpha^\Omega)\phi_\alpha^\Omega))\|_{H^2(\Omega)} \\
&\leq \frac{1}{c_{a,1}}\|L^*v - \sum_\alpha (L^*v, \phi_\alpha^\Omega)\phi_\alpha^\Omega)\|_{L^2(\Omega)} \\
&\leq \frac{2}{c_{a,1}} C_{J,\phi^\Omega} 2^{-j(s-2)}\|L^*v\|_{H^{s-2}(\Omega)} \\
&\leq \frac{2c_{a,2}}{c_{a,1}} C_{J,\phi^\Omega} 2^{-j(s-2)}\|v\|_{H^s(\Omega)}.
\end{aligned}$$

For the fourth inequality, we use $\|u - P_{U_h^\Omega} u\|_{H^2(\Omega)} \leq 2\inf_{u_h \in U_h^\Omega} \|u - u_h\|_{H^2(\Omega)}$. Then (3.10) holds with $C_{J,\theta^\Omega} = \frac{2c_{a,2}}{c_{a,1}} C_{J,\phi^\Omega}$. □

Lemma 3.5. *By definition of U_h^Ω, $Q_{h'}^\Gamma$ and $Q_{h''}^\gamma$, the following Bernstein estimates hold:*

$\forall u \in$ *span* $\{\phi_\alpha^\Omega, \alpha \in K_j\}$

$$\forall 0 < s \leq m, u \in H^s(\Omega) \Rightarrow \|u\|_{H^s(\Omega)} \leq C_{B,\phi^\Omega} 2^{sj}\|u\|_{L^2(\Omega)},$$

$$\forall s < 0, u \in H^s(\Omega) \Rightarrow \|u\|_{H^s(\Omega)} \geq C_{B,\phi^\Omega} 2^{sj}\|u\|_{L^2(\Omega)}.$$

$\forall \lambda \in$ *span* $\{\phi_{\alpha'}^\Gamma, \alpha' \in K'_{j'}\}$

$$\forall 0 < s' \leq m', \lambda \in H^{s'}(\Gamma) \Rightarrow \|\lambda\|_{H^{s'}(\Gamma)} \leq C_{B,\phi^\Gamma} 2^{s'j'}\|\lambda\|_{L^2(\Gamma)},$$

$$\forall s' < 0, \lambda \in H^{s'}(\Gamma) \Rightarrow \|\lambda\|_{H^{s'}(\Gamma)} \geq C_{B,\phi^\Gamma} 2^{s'j'}\|\lambda\|_{L^2(\Gamma)}.$$

$\forall \mu \in$*span* $\{\phi_{\alpha''}^\gamma, \alpha'' \in K''_{j''}\}$

$$\forall 0 < s'' \leq m'', \mu \in H^{s''}(\gamma) \Rightarrow \|\mu\|_{H^{s''}(\gamma)} \leq C_{B,\phi^\gamma} 2^{s''j''}\|\mu\|_{L^2(\gamma)},$$

$$\forall s'' < 0, \mu \in H^{s''}(\gamma) \Rightarrow \|\mu\|_{H^{s''}(\gamma)} \geq C_{B,\phi^\gamma} 2^{s''j''}\|\mu\|_{L^2(\gamma)}.$$

We will give some important inequalities about the bilinear form and trace operator. They will be used in the error estimate and condition number analysis.

Lemma 3.6. *For U_h^Ω, V_h^Ω constructed as previously, if L is coercive on U_h^Ω, then there exist α and β such that the bilinear form $a_\Omega(\cdot,\cdot)$ satisfies*

$$\sup_{\substack{\tilde{y}_h\in V_h^\Omega\\ \tilde{y}_h\neq 0}} \frac{a_\Omega(y_h,\tilde{y}_h)}{\|\tilde{y}_h\|_{H^1(\Omega)}} \geq \alpha\|y_h\|_{H^1(\Omega)}, \quad \forall y_h\in U_h^\Omega, \tag{3.11}$$

$$\sup_{\substack{y_h\in U_h^\Omega\\ y_h\neq 0}} \frac{a_\Omega(y_h,\tilde{y}_h)}{\|y_h\|_{H^1(\Omega)}} \geq \beta\|\tilde{y}_h\|_{H^1(\Omega)}, \quad \forall \tilde{y}_h\in V_h^\Omega. \tag{3.12}$$

Proof.

$$\begin{aligned}
\sup_{\substack{\tilde{y}_h\in V_h^\Omega\\ \tilde{y}_h\neq 0}} \frac{a_\Omega(y_h,\tilde{y}_h)}{\|\tilde{y}_h\|_{H^1(\Omega)}} &= \sup_{\substack{\tilde{y}_h\in V_h^\Omega\\ \tilde{y}_h\neq 0}} \frac{(Ly_h,\tilde{y}_h)}{\|\tilde{y}_h\|_{H^1(\Omega)}}\\
&\geq \sup_{\substack{y\in H^1_{\mathcal{P}}(\Omega)\\ y\neq 0}} \frac{(Ly_h,P_{V_h^\Omega}y)}{\|P_{V_h^\Omega}y\|_{H^1(\Omega)}}.
\end{aligned}$$

According to the definition of $P_{V_h^\Omega}$, we get

$$\begin{aligned}
(Ly_h,P_{V_h^\Omega}y) &= (Ly_h,\sum_\alpha (y,\tilde{\theta}_\alpha^\Omega)\theta_\alpha^\Omega)\\
&= (y_h,\sum_\alpha (y,\tilde{\theta}_\alpha^\Omega)\phi_\alpha^\Omega)\\
&= \sum_\alpha (y,L\phi_\alpha^\Omega)(y_h,\phi_\alpha^\Omega)\\
&= (Ly_h,y).
\end{aligned}$$

Combining the equality above and $\|P_{V_h^\Omega}y\|_{H^1(\Omega)}\leq\|y\|_{H^1(\Omega)}$,

$$\begin{aligned}
\sup_{\substack{\tilde{y}_h\in V_h^\Omega\\ \tilde{y}_h\neq 0}} \frac{a_\Omega(y_h,\tilde{y}_h)}{\|\tilde{y}_h\|_{H^1(\Omega)}} &\geq \sup_{\substack{y\in H^1_{\mathcal{P}}(\Omega)\\ y\neq 0}} \frac{(Ly_h,y)}{\|y\|_{H^1(\Omega)}}\\
&\geq \alpha\|y_h\|_{H^1(\Omega)},
\end{aligned}$$

α is the coercive coefficient.

In the same way, we get the inequality (3.12). □

Then we concern the inequalities satisfied by the trace operator T_Γ and T_γ. They describe the compatibility between V_h^Ω, $Q_{h'}^\Gamma$, and $Q_{h''}^\gamma$, called LBB condition.

Theorem 3.7. [6],[25]*For U_h^Ω, V_h^Ω, $Q_{h'}^\Gamma$ and $Q_{h''}^\gamma$ constructed as previously, if $Q_{h'}^\Gamma \in H^{s'-\frac{1}{2}}(\Gamma)$ with $s' > 2$, $Q_{h''}^\gamma \in H^{s''-\frac{1}{2}}(\gamma)$ with $s'' > 1$, then (3.13) and (3.14) hold under the condition that, there exist two positive constants B_1 and B_2, such that $j(s'-2) - j'(s'-1) > B_1$ and $j - j'' > B_2$.*

$$\sup_{\substack{\tilde{y}_h \in V_h^\Omega \\ \tilde{y}_h \neq 0}} \frac{(T_\Gamma \tilde{y}_h, p_{h'})}{\|\tilde{y}_h\|_{H^1(\Omega)}} \geq \beta_1 \|p_{h'}\|_{H^{-\frac{1}{2}}(\Gamma)}, \quad \forall p_{h'} \in Q_{h'}^\Gamma, \tag{3.13}$$

$$\sup_{\substack{y_h \in U_h^\Omega \\ y_h \neq 0}} \frac{(T_\gamma y_h, \tilde{q}_{h''})}{\|y_h\|_{H^1(\Omega)}} \geq \beta_2 \|\tilde{q}_{h''}\|_{H^{-\frac{1}{2}}(\gamma)}, \quad \forall \tilde{q}_{h''} \in Q_{h''}^\gamma. \tag{3.14}$$

where the constants $\beta_1, \beta_2 > 0$ independent of h.

There exists a variant version of (3.13) which will be useful in the error estimate and conditionment analysis.

Theorem 3.8. *If $Q_{h'}^\Gamma \in H^{s'-\frac{1}{2}}(\Gamma)$ with $s' > 2$, $j(s'-2) - j'(s'-1) > B_1$, B_1 is a constant, then*

$$\sup_{\substack{\tilde{y}_h \in V_h^\Omega \\ \tilde{y}_h \neq 0}} \frac{(T_\Gamma \tilde{y}_h, p_{h'})}{\|\tilde{y}_h\|_{H^2(\Omega)}} \geq \tilde{\beta}_1 \|p_{h'}\|_{H^{-\frac{1}{2}}(\Gamma)}, \quad \forall p_{h'} \in Q_{h'}^\Gamma.$$

where the constant $\tilde{\beta}_1 > 0$ independent of h.

First we recall the lemma in [25],

Lemma 3.9. *For $p_{h'} \in Q_{h'}^\Gamma$, there exists $p_{h'}^* \in Q_{h'}^\Gamma \cap H^{s'-1/2}$, $s' \geq 1$ such that*

$$\|p_{h'}\|_{H^{-1/2}(\Gamma)} \|p_{h'}^*\|_{H^{1/2}(\Gamma)} \leq K \int_\Gamma p_{h'} p_{h'}^* d\sigma,$$

where K is a constant which depends on the problem.

Lemma 3.10. *Let $Q_{h'}^\Gamma \subset H^{s'-\frac{1}{2}}(\Gamma)$ with $s' > 2$, for $p_{h'} \in Q_{h'}^\Gamma$, there exists $y_h \in V_h^\Omega$ such that*

$$\|p_{h'} - T_\Gamma y_h\|_{H^{3/2}(\Gamma)} \leq c_{s,1} c_{s,2} C_{J,\theta^\Omega} C_{B,\phi^\Gamma} 2^{-j(s'-2)+j'(s'-1)} \|p_{h'}\|_{H^{\frac{1}{2}}(\Gamma)}, \tag{3.15}$$

and

$$\|y_h\|_{H^2(\Omega)} \leq c_{s,2} (C_{J,\theta^\Omega} C_{B,\phi^\Gamma} 2^{-j(s'-2)+j'(s'-1)} + C_{B,\phi^\Gamma} 2^{j'}) \|p_{h'}\|_{H^{\frac{1}{2}}(\Gamma)}. \tag{3.16}$$

Proof. Since $p_{h'} \in H^{s'-\frac{1}{2}}(\Gamma)$, $s' > 2$ and trace theorem, there exists $Ep_{h'} \in H^{s'}(\Omega)$ such that

$$T_\Gamma E p_{h'} = p_{h'}, \tag{3.17}$$

and

$$\|Ep_{h'}\|_{H^{s'}(\Omega)} \le c_{,2}\|p_{h'}\|_{H^{s'-\frac{1}{2}}(\Gamma)}.$$

For any $p_{h'} \in Q^\Gamma_{h'}$, pick $y_h \in V^\Omega_h$ such that

$$\|Ep_{h'} - y_h\|_{H^2(\Omega)} = \min_{y\in V^\Omega_h} \|Ep_{h'} - y\|_{H^2(\Omega)}.$$

We infer from the trace theorem and (3.17) that

$$\|Ep_{h'} - y_h\|_{H^2(\Omega)} \ge \frac{1}{c_{s,1}}\|T_\Gamma(Ep_{h'} - y_h)\|_{H^{3/2}(\Gamma)} = \frac{1}{c_{s,1}}\|p_{h'} - T_\Gamma y_h\|_{H^{3/2}(\Gamma)}.$$

Using the Jackson and Bernstein estimate leads to,

$$\begin{aligned}\|p_{h'} - T_\Gamma y_h\|_{H^{3/2}(\Gamma)} &\le c_{s,1}C_{J,\theta^\Omega}2^{-j(s'-2)}\|Ep_{h'}\|_{H^{s'}(\Omega)}\\ &\le c_{s,1}c_{s,2}C_{J,\theta^\Omega}C_{B,\phi^\Gamma}2^{-j(s'-2)+j'(s'-1)}\|p_{h'}\|_{H^{\frac{1}{2}}(\Gamma)}.\end{aligned}$$

Moreover, by the triangle inequality,

$$\begin{aligned}\|y_h\|_{H^2(\Omega)} &\le \|Ep_{h'} - y_h\|_{H^2(\Omega)} + \|Ep_{h'}\|_{H^2(\Omega)}\\ &\le c_{s,2}(C_{J,\theta^\Omega}C_{B,\phi^\Gamma}2^{-j(s'-2)+j'(s'-1)} + C_{B,\phi^\Gamma}2^{j'})\|p_{h'}\|_{H^{\frac{1}{2}}(\Gamma)}.\end{aligned}$$

□

Now we are ready to prove theorem 3.8.

Proof. By lemma, for $p_{h'} \in Q^\Gamma_{h'} \subset H^{s'-\frac{1}{2}}(\Gamma)$, $s > 2$ there exists $\tilde{p}_{h'} \in Q^\Gamma_{h'}$ such that

$$\begin{aligned}\|p_{h'}\|_{H^{-1/2}(\Gamma)}\|\tilde{p}_{h'}\|_{H^{1/2}(\Gamma)} &\le K\int_\Gamma p_{h'}\tilde{p}_{h'}d\sigma\\ &\le K\int_\Gamma p_{h'}(\tilde{p}_{h'} - T_\Gamma y_h)d\sigma + K\int_\Gamma p_{h'}T_\Gamma y_h d\sigma,\end{aligned}$$

for all $y_h \in V^\Omega_h$.

Choose the element $\tilde{y}_h$ in V_h^Ω which satisfies the inequality (3.15), then we have

$$\begin{aligned}\|p_{h'}\|_{H^{-1/2}(\Gamma)}\|\tilde{p}_{h'}\|_{H^{1/2}(\Gamma)} &\leq K\|p_{h'}\|_{H^{-1/2}(\Gamma)}\|\tilde{p}_{h'} - T_\Gamma \tilde{y}_h\|_{H^{1/2}(\Gamma)} + K\int_\Gamma p_{h'} T_\Gamma \tilde{y}_h d\sigma \\ &\leq K c_{s,1} c_{s,2} C_{J,\theta^\Omega} C_{B,\phi^\Gamma} 2^{-j(s'-2)+j'(s'-1)} \|p_{h'}\|_{H^{-1/2}(\Gamma)}\|\tilde{p}_{h'}\|_{H^{1/2}(\Gamma)} \\ &+ K\int_\Gamma p_{h'} T_\Gamma \tilde{y}_h d\sigma.\end{aligned}$$

It writes

$$\|p_{h'}\|_{H^{-1/2}(\Gamma)}\|\tilde{p}_{h'}\|_{H^{1/2}(\Gamma)}(1 - K c_{s,1} c_{s,2} C_{J,\theta^\Omega} C_{B,\phi^\Gamma} 2^{-j(s'-2)+j'(s'-1)}) \leq K\int_\Gamma p_{h'} T_\Gamma \tilde{y}_h d\sigma.$$

Suppose $1 - K c_{s,1} c_{s,2} C_{J,\theta^\Omega} C_{B,\phi^\Gamma} 2^{-j(s'-2)+j'(s'-1)} > 0$, i.e. $j(s'-2) - j'(s'-1) > B_1$, where $B_1 = ln(K c_{s,1} c_{s,2} C_{J,\theta^\Omega} C_{B,\phi^\Gamma})/ln(2)$ and use (3.16)

$$\|p_{h'}\|_{H^{-1/2}(\Gamma)}\|\tilde{y}_h\|_{H^2(\Omega)} \frac{1 - K c_{s,1} c_{s,2} C_{J,\theta^\Omega} C_{B,\phi^\Gamma} 2^{-j(s'-2)+j'(s'-1)}}{K c_{s,2}(C_{J,\theta^\Omega} C_{B,\phi^\Gamma} 2^{-j(s'-2)+j'(s'-1)} + C_{B,\phi^\Gamma} 2^{j'})} \leq \int_\Gamma p_{h'} T_\Gamma \tilde{y}_h d\sigma,$$

hence for all $p_{h'} \in Q_{h'}^\Gamma$

$$\frac{1 - K c_{s,1} c_{s,2} C_{J,\theta^\Omega} C_{B,\phi^\Gamma} 2^{-j(s'-2)+j'(s'-1)}}{K c_{s,2}(C_{J,\theta^\Omega} C_{B,\phi^\Gamma} 2^{-j(s'-2)+j'(s'-1)} + C_{B,\phi^\Gamma} 2^{j'})} \leq \sup_{\substack{\tilde{y}_h \in V_h^\Omega \\ \tilde{y}_h \neq 0}} \frac{(T_\Gamma \tilde{y}_h, p_{h'})}{\|\tilde{y}_h\|_{H^2(\Omega)}\|p_{h'}\|_{H^{-1/2}(\Gamma)}}.$$

□

3.4 Analyse des erreurs

3.4.1 Estimation globale

We first derive an estimate for the norm of the error $u - u_h$ evaluated on the whole domain Ω.

Theorem 3.11. *Let the bilinear form $a_\Omega(\cdot,\cdot)$ be continuous and coercive, $g \in \Phi(H^{-1/2}(\Gamma))$, and the matrix $\mathcal{A}$ be invertible. If (u, λ) (resp. $(u_h, \lambda_{h'})$) is the solution of problem (3.3) (resp. problem (3.7)), if $u \in H^s(\Omega)$,$0 < s \leq n$ and $\lambda \in H^{s'}(\Gamma)$,$0 < s' \leq n'$, then there exists C_1 and C_2 such that*

$$\|u - u_h\|_{L^2(\Omega)} \leq C_1 2^{-js}\|u\|_{H^s(\Omega)} + C_2 2^{-j's'}\|\lambda\|_{H^{s'}(\Gamma)}.$$

Proof. The fact that u (resp u_h) is the solution of problem (3.3) (resp problem (3.7)) leads to

$$a_\Omega(u-u_h,v_h)+(T_\Gamma v_h,\lambda-\lambda_{h'})_\Gamma=0,\quad \forall v_h\in V_h^\Omega, \tag{3.18}$$

$\forall\tilde{v}\in V_h^\Omega$, $\forall\tilde{u}\in U_h^\Omega$, we have

$$\begin{aligned}a_\Omega(u_h-\tilde{u},\tilde{v}) &= a_\Omega(u_h-u,\tilde{v})+a_\Omega(u-\tilde{u},\tilde{v})\\ &= (T_\Gamma\tilde{v},\lambda-\lambda_{h'})_\Gamma+a_\Omega(u-\tilde{u},\tilde{v}).\end{aligned}$$

$\forall\tilde{\lambda}\in Q_{h'}^\Gamma$, thanks to the trace theorem we get

$$\begin{aligned}a_\Omega(u_h-\tilde{u},\tilde{v})+(T_\Gamma\tilde{v},\lambda_{h'}-\tilde{\lambda})_\Gamma &= (T_\Gamma\tilde{v},\lambda-\tilde{\lambda})_\Gamma+a_\Omega(u-\tilde{u},\tilde{v})\\ &\leq c_{s,1}\|\tilde{v}\|_{H^1(\Omega)}\|\lambda-\tilde{\lambda}\|_{H^{-\frac{1}{2}}(\Gamma)}+\|u-\tilde{u}\|_{L^2(\Omega)}\|L^*\tilde{v}\|_{L^2(\Omega)}\\ &\leq \frac{c_{s,1}}{c_{a,1}}\|\lambda-\tilde{\lambda}\|_{H^{-\frac{1}{2}}(\Gamma)}\|L^*\tilde{v}\|_{L^2(\Omega)}\\ &\quad+\|u-\tilde{u}\|_{L^2(\Omega)}\|L^*\tilde{v}\|_{L^2(\Omega)}.\end{aligned}$$

Dividing by $\|L^*\tilde{v}\|_{L^2(\Omega)}$ and choosing $\sup_{\substack{\tilde{v}\in V_h^\Omega\\ \tilde{v}\neq 0}}$, the above equality becomes

$$\sup_{\substack{\tilde{v}\in V_h^\Omega\\ \tilde{v}\neq 0}}\frac{a_\Omega(u_h-\tilde{u},\tilde{v})}{\|L^*\tilde{v}\|_{L^2(\Omega)}}+\sup_{\substack{\tilde{v}\in V_h^\Omega\\ \tilde{v}\neq 0}}\frac{(T_\Gamma\tilde{v},\lambda_{h'}-\tilde{\lambda})_\Gamma}{\|L^*\tilde{v}\|_{L^2(\Omega)}}\leq\frac{c_{s,1}}{c_{a,1}}\|\lambda-\tilde{\lambda}\|_{H^{-\frac{1}{2}}(\Gamma)}+\|u-\tilde{u}\|_{L^2(\Omega)}.$$

Since $\|L^*\tilde{v}\|_{L^2(\Omega)}\sim\|\tilde{v}\|_{H^2(\Omega)}$ and theorem 3.8,

$$\sup_{\substack{\tilde{v}\in V_h^\Omega\\ \tilde{v}\neq 0}}\frac{a_\Omega(u_h-\tilde{u},\tilde{v})}{\|L^*\tilde{v}\|_{L^2(\Omega)}}\leq\frac{c_{s,1}}{c_{a,1}}\|\lambda-\tilde{\lambda}\|_{H^{-\frac{1}{2}}(\Gamma)}+\|u-\tilde{u}\|_{L^2(\Omega)}.$$

For lower bound, choosing $\tilde{v}$ as $(L^*)^{-1}(u_h-\tilde{u})$ leads to

$$\sup_{\substack{\tilde{v}\in V_h^\Omega\\ \tilde{v}\neq 0}}\frac{a_\Omega(u_h-\tilde{u},\tilde{v})}{\|L^*\tilde{v}\|_{L^2(\Omega)}}\geq\|u_h-\tilde{u}\|_{L^2(\Omega)}.$$

So

$$\|u_h-\tilde{u}\|_{L^2(\Omega)}\leq\frac{c_{s,1}}{c_{a,1}}\|\lambda-\tilde{\lambda}\|_{H^{-\frac{1}{2}}(\Gamma)}+\|u-\tilde{u}\|_{L^2(\Omega)}.$$

Using triangle inequality,

$$\|u-u_h\|_{L^2(\Omega)}\leq\|u-\tilde{u}\|_{L^2(\Omega)}+\|u_h-\tilde{u}\|_{L^2(\Omega)}.$$

and taking $\inf_{\tilde{u}\in U_h^\Omega}$,

$$\begin{aligned}\|u-u_h\|_{L^2(\Omega)} &\leq \inf_{\tilde{u}\in U_h^\Omega} 2\|u-\tilde{u}\|_{L^2(\Omega)} + \inf_{\tilde{\lambda}\in H^{-\frac{1}{2}}(\Gamma)} \frac{c_{s,1}}{c_{a,1}}\|\lambda-\tilde{\lambda}\|_{H^{-\frac{1}{2}}(\Gamma)} \\ &\leq 2C_{J,\phi^\Omega}2^{-js}\|u\|_{H^s(\Omega)} + \frac{c_{s,1}C_{J,\phi^\Gamma}}{c_{a,1}}2^{-j's'}\|\lambda\|_{H^{s'}(\Gamma)}.\end{aligned}$$

Noting $C_1 = 2C_{J,\phi^\Omega}$,$C_2 = \frac{c_{s,1}C_{J,\phi^\Gamma}}{c_{a,1}}$,the proof is concluded. □

Remark 3.4.1. When $g \notin \Phi(H^{-1/2}(\Gamma))$, there exists $\tilde{g}$ which approximates it up to arbitrary precision ϵ in norm $H^{-1/2}(\Gamma)$. Assume that $(\hat{u}, \hat{\lambda})$(resp. $(\hat{u}_h, \hat{\lambda}_{h'})$) is the solution of (3.3)(resp. 3.7) with $g = \tilde{g}$. In addition, let u be the solution of the original problem, then we have

$$\|u-\hat{u}_h\|_{L^2(\omega)} \leq c\epsilon + C_1 2^{-js}\|\hat{u}\|_{H^s(\Omega)} + C_2 2^{-j's'}\|\hat{\lambda}\|_{H^{s'}(\Gamma)}.$$

We now evaluate the error on the Lagrange multipliers. We have the following

Theorem 3.12. *Let the bilinear form $a_\Omega(\cdot,\cdot)$ be continuous and coercive, $g \in \Phi(H^{-1/2}(\Gamma))$, and the matrix $\mathcal{A}$ be invertible. If (u,λ) (resp. $(u_h, \lambda_{h'})$) is the solution of problem (3.3)(resp. problem (3.7)), if $u \in H^s(\Omega)$,$0 < s \leq n$ and $\lambda \in H^{s'}(\Gamma)$,$0 < s' \leq n'$, then there exists two constants C_3 and C_4 such that*

$$\|\lambda-\lambda_{h'}\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C_3 2^{-js}\|u\|_{H^s(\Omega)} + C_4 2^{-j's'}\|\lambda\|_{H^{s'}(\Gamma)}.$$

Proof. For $\forall\tilde{\lambda} \in Q_{h'}^\Gamma$, $\forall\tilde{v} \in V_h^\Omega$, from trace theorem we find

$$\begin{aligned}(T_\Gamma\tilde{v}, \tilde{\lambda}-\lambda) &\leq c_{s,1}\|\tilde{v}\|_{H^1(\Omega)}\|\tilde{\lambda}-\lambda\|_{H^{-\frac{1}{2}}(\Gamma)} \\ &\leq c_{s,1}\|\tilde{v}\|_{H^2(\Omega)}\|\tilde{\lambda}-\lambda\|_{H^{-\frac{1}{2}}(\Gamma)}.\end{aligned}$$

using (3.18)

$$\begin{aligned}(T_\Gamma\tilde{v}, \lambda-\lambda_{h'}) &= -a_\Omega(u-u_h, \tilde{v}) \\ &\leq \|u-u_h\|_{L^2(\Omega)}\|L^*\tilde{v}\|_{L^2(\Omega)} \\ &\leq c_{a,2}\|u-u_h\|_{L^2(\Omega)}\|\tilde{v}\|_{H^2(\Omega)}.\end{aligned}$$

From the above two inequalities and theorem 3.8

$$\begin{aligned}
\|\tilde{\lambda}-\lambda_{h'}\|_{H^{-\frac{1}{2}}(\Gamma)} &\leq \frac{1}{\tilde{\beta}_1}\sup_{\substack{\tilde{v}\in V_h^\Omega\\ \tilde{v}\neq 0}}\frac{(T_\Gamma\tilde{v},\tilde{\lambda}-\lambda_{h'})_\Gamma}{\|\tilde{v}\|_{H^2(\Omega)}}\\
&\leq \frac{c_{s,1}}{\tilde{\beta}_1}\|\tilde{\lambda}-\lambda\|_{H^{-\frac{1}{2}}(\Gamma)}+\frac{c_{a,2}}{\tilde{\beta}_1}\|u-u_h\|_{L^2(\Omega)}.
\end{aligned}$$

Triangle inequality, Jackson estimate combined with theorem 3.11 provide

$$\begin{aligned}
\|\lambda-\lambda_{h'}\|_{H^{-\frac{1}{2}}(\Gamma)} &\leq \inf_{\tilde{\lambda}\in Q_{h'}^\Gamma}\|\lambda-\tilde{\lambda}\|_{H^{-\frac{1}{2}}(\Gamma)}+\inf_{\tilde{\lambda}\in Q_{h'}^\Gamma}\|\tilde{\lambda}-\lambda_{h'}\|_{H^{-\frac{1}{2}}(\Gamma)}\\
&\leq (1+\frac{c_{s,1}}{\tilde{\beta}_1})\inf_{\tilde{\lambda}\in Q_{h'}^\Gamma}\|\lambda-\tilde{\lambda}\|_{H^{-\frac{1}{2}}(\Gamma)}+\frac{c_{a,2}}{\tilde{\beta}_1}\|u-u_h\|_{L^2(\Omega)}\\
&\leq \frac{C_1c_{a,2}}{\tilde{\beta}_1}2^{-js}\|u\|_{H^s(\Omega)}+((1+\frac{c_{s,1}}{\tilde{\beta}_1})C_{J,\phi^\Gamma}+\frac{C_2c_{a,2}}{\tilde{\beta}_1})2^{-j's'}\|\lambda\|_{H^{s'}(\Gamma)}\\
&= C_32^{-js}\|u\|_{H^s(\Omega)}+C_42^{-j's'}\|\lambda\|_{H^{s'}(\Gamma)}.
\end{aligned}$$

Here $C_3=\frac{C_1c_{a,2}}{\tilde{\beta}_1}$,$C_4=((1+\frac{c_{s,1}}{\tilde{\beta}_1})C_{J,\phi^\Gamma}+\frac{C_2c_{a,2}}{\tilde{\beta}_1})$. □

Remark 3.4.2. We compare the result above with the classical one in [55]. There the error estimates are obtained as follows:

$$\begin{aligned}
\|u-u_h\|_{H^1(\Omega)} &\leq c(\inf_{\tilde{u}\in U_h^\Omega}\|u-\tilde{u}\|_{H^1(\Omega)}+\inf_{\tilde{\lambda}\in H^{-\frac{1}{2}}(\Gamma)}\|\lambda-\tilde{\lambda}\|_{H^{-\frac{1}{2}}(\Gamma)})\\
&\leq c(2^{-j(s-1)}\|u\|_{H^s(\Omega)}+2^{-j's'}\|\lambda\|_{H^{s'}(\Gamma)}),
\end{aligned}$$

and

$$\begin{aligned}
\|\lambda-\lambda_{h'}\|_{H^{-\frac{1}{2}}(\Gamma)} &\leq c(\inf_{\tilde{u}\in U_h^\Omega}\|u-\tilde{u}\|_{H^1(\Omega)}+\inf_{\tilde{\lambda}\in H^{-\frac{1}{2}}(\Gamma)}\|\lambda-\tilde{\lambda}\|_{H^{-\frac{1}{2}}(\Gamma)})\\
&\leq c(2^{-j(s-1)}\|u\|_{H^s(\Omega)}+2^{-j's'}\|\lambda\|_{H^{s'}(\Gamma)})
\end{aligned}.$$

It offers another form of norm $H^1(\Omega)$ to measure the error convergence rate between u and u_h. For the convergence rate of Lagrange multipliers, it is a worse result compared to ours.

Remark 3.4.3. From the error estimate if the discrete problem is always invertible, one can conclude that (u,λ) is the unique solution of the problem (3.3).

We know that the introduction of Lagrange multipliers implies that the regularity of u on the whole domain Ω is limited by 3/2. It is more interesting to establish a local interior error estimate which depends on the regularity of u on a subdomain of ω.

3.4.2 Estimation intérieure

A first interior estimate derived in the univariate case

We revisit the content of interior estimates in section 2.3.1 and note the difference in our proof. The problem in the univariate case reads:

$$\begin{cases} (I-\nu\dfrac{\partial^2}{\partial x^2})u = f, \text{ on } [a,b] \\ u(a)=\alpha, \quad u(b)=\beta, \end{cases}$$

where $\nu \geq 0$, $[a,b] \subset \Omega = [0,1]$. The functions ϕ and ψ come from a multiresolution of $L^2(I\!R)$, have compact support. This multiresolution is of m regularity and n order. More precisely, there exists a constant N such that for any $k \in Z\!\!Z$

$$\text{supp}\phi_{j_0,k} \subset [(k-N)2^{-j_0},(k+N)2^{-j_0}],$$

$$\text{supp}\psi_{l,k} \subset [(k-N)2^{-l},(k+N)2^{-l}].$$

Since ϕ can not be mapped under the operator $(-\nu\frac{\partial^2}{\partial x^2})^{-1}$, we introduce a new function φ defined as follows:

Lemma 3.13. *If φ is defined by $\hat{\varphi}(\xi) = sin^2(2\pi\xi)\hat{\phi}(\xi)$, then*

$$\varphi(x) = -\frac{1}{4}(\phi(x-2) - 2\phi(x) + \phi(x+2)),$$

Moreover, φ is compactly supported and $\varphi(x)$ has two vanishing moments.

Proof. Note

$$sin^2(2\pi\xi) = (\frac{e^{i2\pi\xi} - e^{-i2\pi\xi}}{2i})^2 = -\frac{1}{4}(e^{4i\pi\xi} - 2 + e^{-4i\pi\xi}).$$

The inverse Fourier transform leads to

$$\varphi(x) = -\frac{1}{4}(\phi(x-2) - 2\phi(x) + \phi(x+2)).$$

From the definition of $\hat{\varphi}$, it comes that

$$\frac{d^r\hat{\varphi}(\xi)}{d\xi} \mid_{\xi=0} = 0, \quad r = 0,1.$$

that is equivalent to the fact that $\varphi(x)$ has two vanishing moments. □

Applying the smooth fictitious domain method with $\Xi = [a', b'] \supset [a, b]$, the continuous problem reads

$$\begin{cases} \text{Find } u \in H^1([0,1]), \lambda = (\lambda_1, \lambda_2) \in I\!R^2, \text{such that} \\ ((I - \nu \frac{\partial^2}{\partial x^2})u, v) + \lambda_1 v(a') + \lambda_2 v(b') = (\tilde{f}, v), \quad \forall v \in H^1([0,1]), \\ u(a) = \alpha, \quad u(b) = \beta, \end{cases}$$

where $\tilde{f}$ is an extension of f on $[0, 1]$. Define

$$\theta_{l,k}(x) = (I - \nu \frac{\partial^2}{\partial x^2})^{-1} \psi_{l,k},$$

$$\vartheta_{j_0,k}(x) = (I - \nu \frac{\partial^2}{\partial x^2})^{-1} \varphi_{j_0,k}.$$

The next lemma [50] states their decrease rates:

Lemma 3.14. *Let $\theta_{l,k}$ and $\vartheta_{j_0,k}$ defined as above, the following inequalities hold:*

$$|2^{2l} \theta_{l,k}(x)| \leq c \frac{2^{l/2}}{(1 + |2^l x - k|)^n}$$

$$|2^{2j_0} \vartheta_{j_0,k}(x)| \leq c \frac{2^{j_0/2}}{(1 + |2^{j_0} x - k|)^2}$$

Using the previously defined functions we get: $\forall l, k$

$$(u, \psi_{l,k}) = (\tilde{f}, \theta_{l,k}) - \lambda_1 \theta_{l,k}(a') - \lambda_2 \theta_{l,k}(b'),$$

$$(u, \varphi_{j_0,k}) = (\tilde{f}, \vartheta_{j_0,k}) - \lambda_1 \vartheta_{j_0,k}(a') - \lambda_2 \vartheta_{j_0,k}(b').$$

The approximation is performed using a wavelet Petrov-Galerkin method using $U_j^{[0,1]}$ and $V_j^{[0,1]}$,

$$U_j^{[0,1]} = \text{span}\{\phi_{j_0,k}, \psi_{l,k}, k \in Z\!\!Z, j_0 \leq l < j\},$$

$$V_j^{[0,1]} = \text{span}\{\vartheta_{j_0,k}, \theta_{l,k}, k \in Z\!\!Z, j_0 \leq l < j\}.$$

Hence we obtain the linear system,

$$\begin{cases} \text{Find } u_h \in U_j^{[0,1]}, \lambda_h = (\lambda_{h,1}, \lambda_{h,2}) \in I\!R^2, \text{such that} \\ ((I - \nu \frac{\partial^2}{\partial x^2})u_h, v_h) + \lambda_{h,1} v_h(a') + \lambda_{h,2} v_h(b') = (\tilde{f}, v_h), \quad \forall v_h \in V_j^{[0,1]}, \\ u_h(a) = \alpha, \quad u_h(b) = \beta. \end{cases}$$

Since the Petrov-Galerkin method used and the basis of $V_j^{[0,1]}$ not compactly supported, the interior estimate is not straightforward. Then we introduce a new problem to approximate the original one. It is discretized by the Petrov-Galerkin method with compactly supported basis.

$$\begin{cases} -\nu\dfrac{\partial^2}{\partial x^2}\bar{u} = \tilde{f}', \text{ on } [a,b] \\ \bar{u}(a) = \alpha, \quad \bar{u}(b) = \beta, \end{cases}$$

where $\hat{\tilde{f}}'(\xi) = \frac{4\pi^2\xi^2\nu\hat{\tilde{f}}(\xi)}{1+4\pi^2\xi^2\nu}$. Applying again the smooth fictitious domain method, the corresponding continuous problem reads:

$$\begin{cases} \text{Find } \bar{u} \in H^1([0,1]), \bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2) \in I\!R^2, \text{such that} \\ (-\nu\dfrac{\partial^2}{\partial x^2}\bar{u}, \bar{v}) + \bar{\lambda}_1\bar{v}(a') + \bar{\lambda}_2\bar{v}(b') = (\tilde{f}', \bar{v}), \quad \forall \bar{v} \in H^1([0,1]), \\ \bar{u}(a) = \alpha, \quad \bar{u}(b) = \beta. \end{cases}$$

Define

$$\tilde{\theta}_{l,k}(x) = (-\nu\frac{\partial^2}{\partial x^2})^{-1}\psi_{l,k},$$
$$\tilde{\vartheta}_{j_0,k}(x) = (-\nu\frac{\partial^2}{\partial x^2})^{-1}\varphi_{j_0,k}.$$

Using the previously defined functions we get: $\forall l, k$

$$(\bar{u}, \psi_{l,k}) = (\tilde{f}, \theta_{l,k}) - \bar{\lambda}_1\tilde{\theta}_{l,k}(a') - \bar{\lambda}_2\tilde{\theta}_{l,k}(b'),$$

$$(\bar{u}, \varphi_{j_0,k}) = (\tilde{f}, \vartheta_{j_0,k}) - \bar{\lambda}_1\tilde{\vartheta}_{j_0,k}(a') - \bar{\lambda}_2\tilde{\vartheta}_{j_0,k}(b'),$$

The approximation is performed using a Petrov-Galerkin method using $U_j^{[0,1]}$ and $\bar{V}_j^{[0,1]}$,

$$\bar{V}_j^{[0,1]} = \text{span}\{\tilde{\vartheta}_{j_0,k}, \tilde{\theta}_{l,k}, k \in Z\!\!Z, j_0 \le l < j\}.$$

Hence we obtain the linear system,

$$\begin{cases} \text{Find } \bar{u}_h \in U_j^{[0,1]}, \bar{\lambda}_h = (\bar{\lambda}_{h,1}, \bar{\lambda}_{h,2}) \in I\!R^2, \text{such that} \\ (-\nu\dfrac{\partial^2}{\partial x^2}\bar{u}_h, \bar{v}_h) + \bar{\lambda}_{h,1}\bar{v}_h(a') + \bar{\lambda}_{h,2}\bar{v}_h(b') = (\tilde{f}', \bar{v}_h), \quad \forall \bar{v}_h \in \bar{V}_j^{[0,1]}, \\ \bar{u}_h(a) = \alpha, \quad \bar{u}_h(b) = \beta, \end{cases}$$

The next lemma concerns the interior error estimate $\bar{u} - \bar{u}_h$ of Petrov-Galerkin method with compactly support basis.

Lemma 3.15. *Let* $\omega_0 \subset \omega_1 \subset [0,1]$, $\bar{u} \in H^1([0,1])$, $\bar{u}_h \in U_j^{[0,1]}$ *and* p *is a nonnegative integer, arbitrary but fixed. If the approximation space* $\bar{V}_j^{[0,1]}$ *satisfies the following conditions.*

Given two arbitrary interval ϖ_0, ϖ, *with* $\varpi_0 \subset \varpi \subset \omega_1$, *then there exists an* m_0 *such that for all* $j_0 \geq m_0$:

(1)There exists a unique solution $u \in \mathring{U}_j^{[0,1]}$ *of*

$$(-\triangle u, v) = 0, \forall v \in \mathring{\bar{V}}_j^{[0,1]}.$$

where $\mathring{U}_j^{[0,1]}$, $\mathring{\bar{V}}_j^{[0,1]}$ *are subspaces of* $U_j^{[0,1]}$, $\bar{V}_j^{[0,1]}$ *respectively and their elements are compactly supported in* ϖ.

(2) For each $u \in H^l(\varpi)$, $\exists \eta \in \bar{V}_j^{[0,1]}$ *such that*

$$\|u - \eta\|_{s,\varpi} \leq c2^{-j(l-s)}\|u\|_{l,\varpi}, \forall 0 \leq s \leq R, s \leq l \leq M$$

(3) Let $w \in C_0^\infty(\varpi_0)$ *and* $u_h \in U_j^{[0,1]}$, *then* $\exists \eta \in \mathring{\bar{V}}_j^{[0,1]}$ *such that*

$$\|wu_h - \eta\|_{1,\varpi} \leq c2^{-j}\|u_h\|_{1,\varpi}$$

Then exists an m_1, *for all* $j_0 > m_1$, *we have*

$$\|\bar{u} - \bar{u}_h\|_{s,\omega_0} \leq c(2^{-j(l-s)}\|\bar{u}\|_{l,\omega_1} + \|\bar{u} - \bar{u}_h\|_{-p,\omega_1}), \quad 1 \leq l \leq M, s = 0, 1$$

Proof. This lemma is deduced from the theorem 5.1 [56] and theorem 2.1 [9]. It is about the interior estimate with Petrov-Galerkin method, compactly supported basis. The proof of (1) mimics the procedure in lemma 3.6 with $L = -\triangle$. (2) From theorem 2.4[61] we know the multiresolution $\bar{V}_j^{[0,1]}$ is of $m+1$ regularity and $n-2$ order. Then applying Jackson estimate, we get the result with $M = n-2$. (3) follows the proof of lemma 3.1, lemma 3.2 in [9] by the norm equivalence of multiresolution $\bar{V}_j^{[0,1]}$ with compactly supported basis. □

Lemma 3.16. *Let u, $\bar{u}$, φ be defined as above, then*

$$\sum_{k\in\mathbb{Z}} |(u-\bar{u},\phi_{j_0,k})|^2 \leq c\sum_{k\in\mathbb{Z}} |(u-\bar{u},\varphi_{j_0,k})|^2.$$

Proof. From the definition of $\varphi_{j_0,k}$, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}}(u-\bar{u},\varphi_{j_0,k})\phi_{j_0,k} &= -\frac{1}{4}\sum_{k\in\mathbb{Z}}(u-\bar{u},\phi_{j_0,k+2}-2\phi_{j_0,k}+\phi_{j_0,k-2})\phi_{j_0,k}\\ &= -\frac{1}{4}\sum_{k\in\mathbb{Z}}(u-\bar{u},\phi_{j_0,k})(\phi_{j_0,k+2}-2\phi_{j_0,k}+\phi_{j_0,k-2})\\ &= \sum_{k\in\mathbb{Z}}(u-\bar{u},\phi_{j_0,k})\varphi_{j_0,k}\\ &= \frac{d^2}{dx^2}\sum_{k\in\mathbb{Z}}(u-\bar{u},\phi_{j_0,k})F_{j_0,k}\end{aligned}$$

where $\frac{d^2}{dx^2}F_{j_0,k} = \varphi_{j_0,k}$ and $\hat{F}_{j_0,k} = -\frac{\hat{\phi}_{j_0,k}(w)sin^2(2\pi w/2^{j_0})}{(2\pi w)^2}$ which is from the definition of generalized derivative, i.e. $\forall f \in C_0^\infty$, $<\frac{d^2}{dx^2}f, F_{j_0,k}> = < f,\varphi_{j_0,k}>$. Note $f_0 = \sum_{k\in\mathbb{Z}}(u-\bar{u},\phi_{j_0,k})F_{j_0,k}$, $f_1 = \frac{d}{dx}\sum_{k\in\mathbb{Z}}(u-\bar{u},\phi_{j_0,k})F_{j_0,k}$. We apply the Poincaré inequality to get

$$\sum_{k\in\mathbb{Z}}|(u-\bar{u},\phi_{j_0,k})|^2 \leq c\|f_0\|^2_{L^2[0,1]} \leq c\|f_1\|^2_{L^2[0,1]} \leq c\|\frac{df_1}{dx}\|^2_{L^2[0,1]} \leq c\sum_{k\in\mathbb{Z}}|(u-\bar{u},\varphi_{j_0,k})|^2$$

where we used the facts that $\{F_{j_0,k}\}_k$ is a Riesz basis(theorem 2.2[61]) and $\{\phi_{j_0,k}\}_k$ is an orthonormal basis. □

Finally, we are ready to proof the main theorem in this section.

Theorem 3.17. *For $U_j^{[0,1]}$, $V_j^{[0,1]}$ and $\bar{V}_j^{[0,1]}$ constructed as above, let $\omega_0 \subset \omega_1 \subset [0,1]$, the multiresolution $U_j^{[0,1]}$ is of order $n \geq 3$. For any p a non negative integer, there exists m_1 such that for all $j_0 > m_1$, we have estimate,*

$$\|u-u_h\|_{1,\omega_0} \leq c(2^{-j(l-1)}\|\bar{u}\|_{l,\omega_1} + \|\bar{u}-\bar{u}_h\|_{-p,\omega_1} + 2^{-3j_0}d^{-2}),$$

where $d = dist\{\omega_0,[a',b']\}$, $l \leq n-2$.

Proof. From triangle inequality, we have

$$\|u-u_h\|_{1,\omega_0} \leq \|u-\bar{u}\|_{1,\omega_0} + \|\bar{u}-\bar{u}_h\|_{1,\omega_0} + \|\bar{u}_h-u_h\|_{1,\omega_0}.$$

The first term on the right side with norm equivalence is bounded by,

$$\begin{aligned}\|u-\bar{u}\|_{1,\omega_0} &\le c(\sum_k |(u-\bar{u},\phi_{j_0,k})|^2 + \sum_{j\ge j_0}\sum_k 2^{2j}|(u-\bar{u},\psi_{j,k})|^2)^{1/2}\\ &\le c(\sum_k |-\lambda_1\vartheta_{j_0,k}(a')-\lambda_2\vartheta_{j_0,k}(b')+\bar{\lambda}_1\tilde{\vartheta}_{j_0,k}(a')+\bar{\lambda}_2\tilde{\vartheta}_{j_0,k}(b')|^2\\ &+\sum_{j\ge j_0}\sum_k 2^{2j}|-\lambda_1\theta_{j,k}(a')-\lambda_2\theta_{j,k}(b')+\bar{\lambda}_1\tilde{\theta}_{j,k}(a')+\bar{\lambda}_2\tilde{\theta}_{j,k}(b')|^2)^{1/2}\end{aligned}$$

As ϕ and ψ are of compact support, the index k in the above inequations are limited. For j_0 great enough, the point values of $\tilde{\vartheta}_{j_0,k}(a')$, $\tilde{\vartheta}_{j_0,k}(b')$, $\tilde{\theta}_{j,k}(a')$ and $\tilde{\theta}_{j,k}(b')$ are 0. For the point value of $\vartheta_{j_0,k}$ and $\theta_{j,k}$, we use lemma 3.14 to get

$$\begin{aligned}\|u-\bar{u}\|_{1,\omega_0} &\le c(\sum_k |\frac{2^{j_0/2}}{2^{4j_0}d^2}|^2 + \sum_{j\ge j_0}\sum_k 2^{2j}|\frac{2^{j/2}}{2^{(n+2)j}d^2}|^2)^{1/2}\\ &\le c2^{-3j_0}d^{-2}\end{aligned}$$

where $d = \text{dist}\{\omega_0, [a', b']\}$.

The bound of the third term $\|\bar{u}_h - u_h\|_{1,\omega_0}$ is obtained in the same way.

Combined with lemma 3.15, we have

$$\|u-u_h\|_{1,\omega_0} \le c(2^{-j(l-1)}\|\bar{u}\|_{l,\omega_1} + \|\bar{u}-\bar{u}_h\|_{-p,\omega_1} + 2^{-3j_0}d^{-2})$$

□

Remark 3.4.4. The last term in the right hand side of the estimate of theorem 3.17 illustrates that the farther is the control boundary $[a', b']$ from the original domain ω, the better is the convergence rate.

Remark 3.4.5. Following the same track as above we get

$$\|u-u_h\|_{0,\omega_0} \le c(2^{-jl}\|\bar{u}\|_{l,\omega_1} + \|\bar{u}-\bar{u}_h\|_{-p,\omega_1} + 2^{-3j_0}d^{-2})$$

Remark 3.4.6. For the bivariate case, the track of the above proof can not be followed. Indeed point values are replaced by trace and scalar product on the control boundary involving the matrix C(see in section 3.5.2). The generalization of theorem 3.17 to bivariate situation is still ongoing research.

Another interior error estimate available in general

Theorem 3.18. *Let $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$, a_Ω be coercive on $H^1(\Omega)$ with coercive coefficient $\alpha > 1$, u (resp u_h) be the solution of problem (3.3) (resp problem (3.7)), $u \in H^l(\Omega_1)$, $0 < l \leq n$, $\lambda \in H^{s'}(\Gamma)$, $0 < s' \leq n'$, then*

$$\|u - u_h\|_{1,\Omega_0} \leq c(2^{-j(l-1)}\|u\|_{l,\Omega_1} + 2^{-j's'}\|\lambda\|_{H^{s'}(\Gamma)}).$$

Proof. Let $\Omega_0 \subset\subset \tilde{\Omega} \subset\subset \Omega_1$, $w \in C_0^\infty(\tilde{\Omega})$ with $w = 1$ on Ω_0 and set $\tilde{u} = wu$. Pick $\tilde{\lambda}$ satisfying

$$\|\tilde{\lambda} - \lambda\|_{-1/2,\Gamma} = \inf_{\mathcal{B}\in Q^\Gamma_{h'}} \|\mathcal{B} - \lambda\|_{-1/2,\Gamma}.$$

Let $T\tilde{u} \in U_h^\Omega$ be the solution of

$$a_\Omega(\tilde{u} - T\tilde{u}, v_h) = -(\mathcal{B}v_h, \lambda - \tilde{\lambda})_\Gamma, \quad \forall v_h \in V_h^\Omega. \tag{3.19}$$

Then

$$\begin{aligned}\|\tilde{u} - T\tilde{u}\|_{1,\Omega} &\leq c \sup_{v\in H^1(\Omega)} \frac{a_\Omega(\tilde{u} - T\tilde{u}, v)}{\|v\|_{1,\Omega}} \\ &\leq c \sup_{v\in H^1(\Omega)} \left(\frac{a_\Omega(\tilde{u} - T\tilde{u}, v - P^*v)}{\|v\|_{1,\Omega}} - \frac{(\mathcal{B}P^*v, \lambda - \tilde{\lambda})_\Gamma}{\|v\|_{1,\Omega}}\right),\end{aligned}$$

where we use (3.19), $P^*v \in V_h^\Omega$ satisfying

$$a_\Omega(u_h, v - P^*v) = 0, \quad \forall u_h \in U_h^\Omega,$$

and

$$\begin{aligned}\|P^*v\|_{1,\Omega} &\leq c \sup_{\varphi\in U_h^\Omega} \frac{a_\Omega(\varphi, P^*v)}{\|\varphi\|_{1,\Omega}} \\ &= c \sup_{\varphi\in U_h^\Omega} \frac{a_\Omega(\varphi, v)}{\|\varphi\|_{1,\Omega}} \\ &\leq c\|v\|_{1,\Omega}.\end{aligned}$$

For $\forall \eta \in U_h^\Omega$,

$$a_\Omega(\tilde{u} - T\tilde{u}, v - P^*v) = a_\Omega(\tilde{u} - \eta, v - P^*v) \leq c\|\tilde{u} - \eta\|_{1,\Omega}\|v\|_{1,\Omega}.$$

If we take $\inf_{\eta\in U_h^\Omega}$ in the right hand side term, then

$$a_\Omega(\tilde{u}-T\tilde{u},v-P^*v)\le c2^{-j(l-1)}\|\tilde{u}\|_{l,\Omega_1}\|v\|_{1,\Omega}\le c2^{-j(l-1)}\|u\|_{l,\Omega_1}\|v\|_{1,\Omega}.$$

We also have with trace theorem and Jackson estimate,

$$|(\mathcal{B}P^*v,\lambda-\tilde{\lambda})_\Gamma|\le c\|P^*v\|_{1,\Omega}\|\lambda-\tilde{\lambda}\|_{-1/2,\Gamma}$$
$$c\|v\|_{1,\Omega}2^{-j's'}\|\lambda\|_{H^{s'}(\Gamma)}.$$

so

$$\|\tilde{u}-T\tilde{u}\|_{1,\Omega}\le c(2^{-j(l-1)}\|u\|_{l,\Omega_1}+2^{-j's'}\|\lambda\|_{H^{s'}(\Gamma)}).$$

Subtracting (3.19) and (3.18) leads to

$$a_\Omega(T\tilde{u}-u_h,v_h)=a_\Omega(\tilde{u}-u,v_h)+(\mathcal{B}v_h,\lambda_{h'}-\tilde{\lambda})_\Gamma,\quad\forall v_h\in V_h^\Omega.$$

Using this equality with trace theorem, (3.18), we get

$$\begin{aligned}\|T\tilde{u}-u_h\|_{1,\Omega}&\le\frac{1}{\alpha}\sup_{v_h\in V_h^\Omega}\frac{a_\Omega(T\tilde{u}-u_h,v_h)}{\|v_h\|_{1,\Omega}}=\frac{1}{\alpha}\sup_{v_h\in V_h^\Omega}\frac{a_\Omega(\tilde{u}-u,v_h)+(\mathcal{B}v_h,\lambda_{h'}-\tilde{\lambda})_\Gamma}{\|v_h\|_{1,\Omega}}\\&\le\frac{1}{\alpha}(\sup_{v_h\in V_h^\Omega}\frac{a_\Omega(\tilde{u}-T\tilde{u},v_h)}{\|v_h\|_{1,\Omega}}+\sup_{v_h\in V_h^\Omega}\frac{a_\Omega(T\tilde{u}-u_h,v_h)}{\|v_h\|_{1,\Omega}}+\sup_{v_h\in V_h^\Omega}\frac{(\mathcal{B}v_h,\lambda-\tilde{\lambda})_\Gamma}{\|v_h\|_{1,\Omega}}).\end{aligned}$$

From the Jackson estimate, we have

$$\|T\tilde{u}-u_h\|_{1,\Omega}\le\frac{1}{\alpha}(\|\tilde{u}-T\tilde{u}\|_{1,\Omega}+\|T\tilde{u}-u_h\|_{1,\Omega}+c_{s,1}2^{-j's'}\|\lambda\|_{H^{s'}(\Gamma)}),$$

hence

$$\|T\tilde{u}-u_h\|_{1,\Omega}\le\frac{1}{\alpha-1}(\|\tilde{u}-T\tilde{u}\|_{1,\Omega}+c_{s,1}2^{-j's'}\|\lambda\|_{H^{s'}(\Gamma)}).$$

So

$$\begin{aligned}\|u-u_h\|_{1,\Omega_0}&\le\|u-T\tilde{u}\|_{1,\Omega_0}+\|T\tilde{u}-u_h\|_{1,\Omega_0}\\&\le\|\tilde{u}-T\tilde{u}\|_{1,\Omega}+\|T\tilde{u}-u_h\|_{1,\Omega}\\&\le c(2^{-j(l-1)}\|u\|_{l,\Omega_1}+2^{-j's'}\|\lambda\|_{H^{s'}(\Gamma)}).\end{aligned}$$

□

Remark 3.4.7. Here the last term in the right hand side term of the estimates of theorem 3.18 illustrates the influence of the Lagrange multiplier and therefore the amplitude of the normal gradient jump at the boundary.

3.5 Analyse du conditionnement de $(DC)^tDC$

3.5.1 Cas unidimensionnels

We first consider the univariate situation where for $0 < a' < a < b < b' < 1$, $\omega =]a, b[$, $\Xi =]a', b'[$, $\Omega =]0, 1[$. We write $u_h = \sum_{k=0}^{2^j-1} (U_h)_k \phi_k^j$ and $\Lambda_{h'} = (\lambda_1, \lambda_2)$, with ϕ_k^j the periodic spline scaling function of order m on the interval $[0, 1]$ and $\theta_k^j = (L^*)^{-1}\phi_k^j$.

$$(F_h)_k = \int_0^1 \tilde{f}(x)\theta_k^j(x)dx, \tag{3.20}$$

with $\tilde{f}$ defined as an extension of f on $[0, 1]$,

$$C = \begin{pmatrix} \theta_0^j(a') & \theta_0^j(b') \\ \theta_1^j(a') & \theta_1^j(b') \\ \vdots & \vdots \\ \theta_{2^j-1}^j(a') & \theta_{2^j-1}^j(b') \end{pmatrix},$$

$$D = \begin{pmatrix} \phi_0^j(a) & \phi_1^j(a) & \cdots & \phi_{2^j-1}^j(a) \\ \phi_0^j(b) & \phi_1^j(b) & \cdots & \phi_{2^j-1}^j(b) \end{pmatrix},$$

and

$$G_h = \begin{pmatrix} u(a) \\ u(b) \end{pmatrix}.$$

Using Uzawa algorithm, the problem (3.8) is rewritten as

$$\begin{cases} \text{Find } U_h \in I\!R^{2^j}, \Lambda_{h'} \in I\!R^2 \text{ ,such that} \\ (DC)^tDC\Lambda_{h'} = (DC)^tDF_h - (DC)^tG_h, \\ U_h = F_h - C\Lambda_{h'}. \end{cases}$$

The condition number of $(DC)^tDC$ controls the convergence of the algorithms. Before the analysie of the condition number, we introduce an embedding theorem in 1D which will be used in the proof.

Lemma 3.19. *Let $[a, b]$ be a bounded interval. Then the following inequality holds*

$$max_{a\le x\le b}|v(x)| \le (\frac{1}{b-a} + 2)^{1/2}\|v\|_0^{1/2}\|v\|_1^{1/2}, \quad \forall v \in H^1[a, b].$$

Theorem 3.20. *Assume that the matrix $\mathcal{A}$ is invertible. The* (2×2) *matrix* $(DC)^t(DC)$ *satisfies the following estimate,* $\exists 0 < K_1, K_2 < +\infty$ *such that* $\forall w = (w_1, w_2) \in I\!R^2$

$$K_1(w,w)_{l^2} \leq ((DC)^t(DC)w, w) \leq K_2(w,w)_{l^2}. \tag{3.21}$$

and therefore, $cond_2((DC)^t(DC)) \leq \frac{K_2}{K_1}$.

Proof. • Estimate for C^tC.

By the definition of C, we have with $w = (w_1, w_2) \in I\!R^2$

$$(C^TCw, w) = \sum_{k=0}^{2^j-1} (\theta_k^j(a')w_1 + \theta_k^j(b')w_2)^2.$$

The following inequality follows from theorem 3.8 with $V_h = \sum_k z_k \theta_k^j$, $z = (z_k)^t \in I\!R^{2^j}$

$$\sup_{\substack{V_h \in V_h^\Omega \\ V_h \neq 0}} \frac{w_1 V_h(a') + w_2 V_h(b')}{\|V_h\|_{H^2[0,1]}} \geq \tilde{\beta}_1 \sqrt{w_1^2 + w_2^2}.$$

Then we use the property of operator L^* to get the lower bound of $\|V_h\|_{H^2[0,1]}$

$$\|V_h\|_{H^2[0,1]} \geq \frac{1}{c_{a,2}} \|\sum_k z_k \phi_k^j\|_{L^2[0,1]} = \frac{1}{c_{a,2}} (z,z)_{l^2}^{1/2}.$$

And the upper bound of $w_1 V_h(a') + w_2 V_h(b')$ is derived with the Cauchy-Schwartz inequality

$$\begin{aligned} w_1 V_h(a') + w_2 V_h(b') &= \sum_k z_k (w_1 \theta_k^j(a') + w_2 \theta_k^j(b')) \\ &\leq (z,z)_{l^2}^{1/2} (\sum_k (\theta_k^j(a')w_1 + \theta_k^j(b')w_2)^2)^{1/2} \\ &= (z,z)_{l^2}^{1/2} (C^TCw, w)^{1/2}. \end{aligned}$$

so

$$(C^TCw, w)_{l^2} \geq K_{1,c}(w,w)_{l^2},$$

where $K_{1,c} = (\tilde{\beta}_1 / c_{a,2})^2$.

Taking $z_k = w_1\theta_k^j(a') + w_2\theta_k^j(b')$, we use lemma 3.19 and the property of operator L^*:

$$\begin{aligned}
(C^TCw, w) &= w_1V_h(a') + w_2V_h(b') \\
&\leq (w,w)_{l^2}^{1/2}\sqrt{|V_h(a')|^2 + |V_h(b')|^2} \\
&\leq \sqrt{2}(w,w)_{l^2}^{1/2} \max_{a'\leq x\leq b'} |V_h(x)| \\
&\leq \sqrt{2}(\frac{1}{b'-a'} + 2)^{1/2}(w,w)_{l^2}^{1/2}\|V_h\|_{L^2[a',b']}^{1/2}\|V_h\|_{H^1[a',b']}^{1/2} \\
&\leq \sqrt{2}(\frac{1}{b'-a'} + 2)^{1/2}(w,w)_{l^2}^{1/2}\|V_h\|_{H^2[0,1]} \\
&\leq \frac{\sqrt{2}(\frac{1}{b'-a'} + 2)^{1/2}}{c_{a,1}}(w,w)_{l^2}^{1/2}\|\sum_k z_k\phi_k^j\|_{L^2[0,1]} \\
&= \frac{\sqrt{2}(\frac{1}{b'-a'} + 2)^{1/2}}{c_{a,1}}(w,w)_{l^2}^{1/2}(z,z)_{l^2}^{1/2}.
\end{aligned}$$

Then

$$(C^TCw, w) \leq K_{2,c}(w,w)_{l^2},$$

with $K_{2,c} = 2(\frac{1}{b'-a'} + 2)/c_{a,1}^2$.

- Estimate for D^tD.

We write using $z = (z_k)^t \in I\!R^{2^j}$

$$(D^tDz, z) = (\sum_k z_k\phi_k^j(a))^2 + (\sum_k z_k\phi_k^j(b))^2.$$

By (3.14) and the proposition 7.4.1 in [63], the following inequality holds with $U_h = \sum_k z_k\phi_k^j$, $z = (z_k)_t \in I\!R^{2^j}$ and $\lambda = (\lambda_1, \lambda_2) \in I\!R^2$

$$\sup_{\substack{\lambda\in I\!R^2\\ \lambda\neq 0}} \frac{\lambda_1U_h(a) + \lambda_2U_h(b)}{\sqrt{\lambda_1^2+\lambda_2^2}} \geq \beta_2\|U_h\|_{H^1[0,1]}, \quad \forall z \in (kerD)^\perp.$$

The lower and upper bound respectively for $\|U_h\|_{H^1[0,1]}$ and $\lambda_1U_h(a) + \lambda_2U_h(b)$ are,

$$\|U_h\|_{H^1[0,1]} \geq \|U_h\|_{L^2[0,1]} = (z,z)_{l^2}^{1/2},$$

$$\lambda_1U_h(a) + \lambda_2U_h(b) \leq \sqrt{\lambda_1^2+\lambda_2^2}(D^tDz,z)^{1/2}.$$

Then we get

$$(D^tDz, z) \geq K_{1,d}(z,z)_{l^2}, \quad \forall z \in (kerD)^\perp,$$

with $K_{1,d} = (\beta_2)^2$.

Using again lemma 3.19 we get

$$\begin{aligned}(D^t Dz, z) &= |U_h(a)|^2 + |U_h(b)|^2 \\ &\leq 2 \max_{a\leq x\leq b} |U_h(x)|^2 \\ &\leq 2(\frac{1}{b-a} + 2)\|U_h\|_{L^2[a,b]}\|U_h\|_{H^1[a,b]} \\ &\leq 2(\frac{1}{b-a} + 2)\|U_h\|_{L^2[0,1]}\|U_h\|_{H^1[0,1]} \\ &= 2(\frac{1}{b-a} + 2)(z,z)_{l^2}^{1/2}\|U_h\|_{H^1[0,1]},\end{aligned}$$

From above, we know

$$\|U_h\|_{H^1[0,1]} \leq \frac{1}{\beta_2}(D^t Dz, z)^{1/2}, \quad \forall z \in (kerD)^\perp.$$

it follows that

$$(D^t Dz, z) \leq K_{2,d}(z,z)_{l^2}, \quad \forall z \in (kerD)^\perp,$$

where $K_{2,d} = (\frac{2(\frac{1}{b-a}+2)}{\beta_2})^2$.

• Estimate for $(DC)^t(DC)$.

For any $w \in I\!R^2$, we define $z = Cw$.

If $z \in kerD$, since the matrix $\mathcal{A}$ is invertible, then $z \in \mathrm{ker}D \cap \mathcal{R}(C) = \{0\}$. Moreover C is of full column rank, we get $w = 0$. So (3.21) is true.

If $z \in (kerD)^\perp$, we use the estimates for C^tC and D^tD to conclude the proof of the theorem with $K_1 = K_{1,c}K_{1,d}$ and $K_2 = K_{2,c}K_{2,d}$. The definition of cond_2 gives directly $\mathrm{cond}_2((DC)^t(DC)) \leq \frac{K_2}{K_1} = \frac{8c_{a,2}^2(\frac{1}{b-a}+2)^2(\frac{1}{b'-a'}+2)}{c_{a,1}^2(\beta_2)^4(\tilde{\beta}_1)^2}$. □

3.5.2 Cas bidimensionnels

Here

$$(F_h)_\alpha = (\tilde{f}, \theta_\alpha^\Omega)_{L^2(\Omega)},$$

$$C_{\alpha\alpha'} = (T_\Gamma\theta_\alpha^\Omega, \phi_{\alpha'}^\Gamma)_{L^2(\Gamma)},$$

$$D_{\alpha''\alpha} = (T_\gamma\phi_\alpha^\Omega, \phi_{\alpha''}^\gamma)_{L^2(\gamma)},$$

$$(G_h)_{\alpha''} = (g, \phi^{\gamma}_{\alpha''})_{L^2(\gamma)}.$$

We use the Uzawa algorithm for (5.7), then it is equivalent to solve the following system

$$\begin{cases} \text{Find } U_h \in I\!R^{4^j}, \Lambda_{h'} \in I\!R^{2^{j'}}, \text{ such that} \\ (DC)^t DC\Lambda_{h'} = (DC)^t DF_h - (DC)^t G_h, \\ U_h = F_h - C\Lambda_{h'}. \end{cases}$$

Theorem 3.21. *Assume that the matrix $\mathcal{A}$ is invertible. The $(2^{j'} \times 2^{j'})$ matrix $(DC)^t(DC)$ satisfies the following estimate, $\exists 0 < K_1, K_2 < +\infty$ such that $\forall w \in I\!R^{2^{j'}}$*

$$K_1 2^{-j'-j''}(w,w)_{l^2} \leq ((DC)^t(DC)w, w) \leq K_2 2^{2j}(w,w)_{l^2},$$

where $K_1 = K_{1,c}K_{1,d}$, $K_2 = K_{2,c}K_{2,d}$, $K_{1,c} = (\tilde{\beta}_1 C_{B,\phi^\Gamma}/c_{a,2})^2$, $K_{1,d} = (\beta_2 C_{B,\phi^\gamma})^2$, $K_{2,c} = (\frac{c_{s,1}}{c_{a,1}})^2$, $K_{2,d} = (c_{s,1}C_{B,\phi^\Omega})^2$. And therefore $cond_2((DC)^t DC) \leq \frac{K_2}{K_1} 2^{2j+j'+j''}$.

Proof. • Estimate for $C^t C$.

$\forall w \in I\!R^{2^{j'}}$, according to the definition of matrix C, we have

$$(C^t Cw, w) = \sum_{\alpha\in k_j} |\int_\Gamma T_\Gamma \theta^\Omega_\alpha \sum_{\alpha'\in k'_{j'}} w_{\alpha'}\phi^\Gamma_{\alpha'}|^2.$$

Taking $\tilde{y}_h = \sum_{\alpha\in k_j} z_\alpha \theta^\Omega_\alpha$, $z = (z_\alpha)^t_\alpha \in I\!R^{4^j}$, $p_{h'} = \sum_{\alpha'\in k'_{j'}} w_{\alpha'}\phi^\Gamma_{\alpha'}$ in (3.13) and noting theorem 3.8,

$$\sup_{\substack{z\in I\!R^{4^j}\\ z\neq 0}} \frac{\int_\Gamma \sum_{\alpha\in k_j} z_\alpha T_\Gamma \theta^\Omega_\alpha \sum_{\alpha'\in k'_{j'}} w_{\alpha'}\phi^\Gamma_{\alpha'}}{\|\sum_{\alpha\in k_j} z_\alpha \theta^\Omega_\alpha\|_{H^2(\Omega)}} \geq \tilde{\beta}_1 \|\sum_{\alpha'\in k'_{j'}} w_{\alpha'}\phi^\Gamma_{\alpha'}\|_{H^{-1/2}(\Gamma)}$$
$$\geq \tilde{\beta}_1 C_{B,\phi^\Gamma} 2^{-j'/2}(w,w)^{1/2}_{l^2},$$

the second inequality is from the Bernstein estimate. Moreover we have

$$\int_\Gamma \sum_{\alpha\in k_j} z_\alpha T_\Gamma \theta^\Omega_\alpha \sum_{\alpha'\in k'_{j'}} w_{\alpha'}\phi^\Gamma_{\alpha'} \leq (z,z)^{1/2}_{l^2}(C^t Cw, w)^{1/2},$$

and

$$\|\sum_{\alpha\in k_j} z_\alpha \theta_\alpha^\Omega\|_{H^2(\Omega)} \geq \frac{1}{c_{a,2}}\|\sum_{\alpha\in k_j} z_\alpha \phi_\alpha^\Omega\|_{L^2(\Omega)} = \frac{1}{c_{a,2}}(z,z)_{l^2}^{1/2}.$$

Combining the inequalities above, it derives

$$(C^tCw,w) \geq K_{1,c}2^{-j'}(w,w)_{l^2},$$

with $K_{1,c} = (\frac{\tilde{\beta}_1 C_{B,\phi\Gamma}}{c_{a,2}})^2$.

If we take $z_\alpha = \int_\Gamma T_\Gamma \theta_\alpha^\Omega \sum_{\alpha'\in k'_{j'}} w_{\alpha'}\phi_{\alpha'}^\Gamma$, then

$$\begin{aligned}(C^tCw,w) &= \sum_{\alpha\in k_j} z_\alpha \int_\Gamma T_\Gamma \theta_\alpha^\Omega \sum_{\alpha'\in k'_{j'}} w_{\alpha'}\phi_{\alpha'}^\Gamma \\ &\leq \|\sum_{\alpha\in k_j} z_\alpha T_\Gamma \theta_\alpha^\Omega\|_{L^2(\Gamma)}\|\sum_{\alpha'\in k'_{j'}} w_{\alpha'}\phi_{\alpha'}^\Gamma\|_{L^2(\Gamma)} \\ &\leq c_{s,1}\|\sum_{\alpha\in k_j} z_\alpha \theta_\alpha^\Omega\|_{H^2(\Gamma)}(w,w)_{l^2}^{1/2} \\ &\leq \frac{c_{s,1}}{c_{a,1}}(z,z)_{l^2}^{1/2}(w,w)_{l^2}^{1/2},\end{aligned}$$

with $c_{s,1}$ the trace coefficient. So

$$(C^tCw,w) \leq K_{2,c}(w,w)_{l^2},$$

with $K_{2,c} = (\frac{c_{s,1}}{c_{a,1}})^2$.

- Estimate for D^tD.

$\forall z = (z_\alpha)_\alpha^t \in I\!R^{4^j}$,

$$(D^tDz,z) = \sum_{\alpha''\in k''_{j''}} |\int_\gamma \phi_{\alpha''}^\gamma \sum_{\alpha\in k_j} z_\alpha T_\gamma \phi_\alpha^\Omega|^2.$$

Taking $y_h = \sum_{\alpha\in k_j} z_\alpha\phi_\alpha^\Omega$, $\tilde{q}_{h''} = \sum_{\alpha''\in k''_{j''}} w_{\alpha''}\phi_{\alpha''}^\gamma$, $w = (w_{\alpha''})_{\alpha''}^t \in I\!R^{2^{j''}}$ in (3.14) and using its equivalent form

$$\begin{aligned}\sup_{\substack{w\in I\!R^{2^{j''}}\\ w\neq 0}} \frac{\int_\gamma \sum_{\alpha\in k_j} z_\alpha T_\gamma\phi_\alpha^\Omega \sum_{\alpha''\in k''_{j''}} w_{\alpha''}\phi_{\alpha''}^\gamma}{\|\sum_{\alpha''\in k''_{j''}} w_{\alpha''}\phi_{\alpha''}^\gamma\|_{H^{-1/2}(\gamma)}} &\geq \beta_2\|\sum_{\alpha\in k_j} z_\alpha\phi_\alpha^\Omega\|_{H^1(\Omega)} \\ &\geq \beta_2(z,z)_{l^2}^{1/2}, \quad \forall z\in (kerD)^\perp.\end{aligned}$$

As

$$\int_\gamma \sum_{\alpha\in k_j} z_\alpha T_\gamma \phi_\alpha^\Omega \sum_{\alpha''\in k''_{j''}} w_{\alpha''}\phi^\gamma_{\alpha''} \leq (w,w)^{1/2}_{l^2}(D^tDz,z)^{1/2},$$

and

$$\| \sum_{\alpha''\in k''_{j''}} w_{\alpha''}\phi^\gamma_{\alpha''}\|_{H^{-1/2}(\gamma)} \geq C_{B,\phi^\gamma} 2^{-j''/2}(w,w)^{1/2}_{l^2}.$$

It follows that

$$(D^tDz,z) \geq K_{1,d}2^{-j''}(z,z)_{l^2},$$

with $K_{1,d} = (\beta_2 C_{B,\phi^\gamma})^2$.

If we take $w_{\alpha''} = \int_\gamma \phi^\gamma_{\alpha''} \sum_{\alpha\in k_j} z_\alpha T_\gamma \phi^\Omega_\alpha$,

$$\begin{aligned}(D^tDz,z) &= \int_\gamma \sum_{\alpha''\in k''_{j''}} w_{\alpha''}\phi^\gamma_{\alpha''} \sum_{\alpha\in k_j} z_\alpha T_\gamma\phi^\Omega_\alpha \\ &\leq \| \sum_{\alpha''\in k''_{j''}} w_{\alpha''}\phi^\gamma_{\alpha''}\|_{H^{-1/2}(\gamma)} \| \sum_{\alpha\in k_j} z_\alpha T_\gamma\phi^\Omega_\alpha\|_{H^{1/2}(\gamma)} \\ &\leq (w,w)^{1/2}_{l^2} c_{s,1} \| \sum_{\alpha\in k_j} z_\alpha \phi^\Omega_\alpha\|_{H^1(\Omega)} \\ &\leq (w,w)^{1/2}_{l^2} c_{s,1} C_{B,\phi^\Omega} 2^j (z,z)^{1/2}_{l^2},\end{aligned}$$

then we have

$$(D^tDz,z) \leq K_{2,d}2^{2j}(z,z)_{l^2},$$

with $K_{2,d} = (c_{s,1}C_{B,\phi^\Omega})^2$.

- Estimate for $(DC)^t(DC)$.

It follows as in the univariate case and the proof is completed. □

Diagonal preconditioning

The use of wavelet bases allows to derive diagonal preconditioners. Indeed it follows directly from the above estimate that if we set $\tilde{C} = 2^{-\frac{|\alpha|+|\alpha'|}{2}}(T_\Gamma \theta^\Omega_\alpha, \phi^\Gamma_{\alpha'})_{L^2(\Gamma)}$ and $\tilde{D} = 2^{-\frac{|\alpha|+|\alpha''|}{2}}(T_\gamma\phi^\Omega_\alpha, \phi^\gamma_{\alpha''})_{L^2(\gamma)}$, then the condition number of $(\tilde{D}\tilde{C})^t(\tilde{D}\tilde{C})$ is independent of the parameters j, j', j'', i.e. $\mathrm{Cond}_2(\tilde{D}\tilde{C})^t(\tilde{D}\tilde{C}) \leq K$.

3.6 Implémentation numérique

3.6.1 Cas unidimensionnels

In all the implementation except for specific cases, $L = I - \triangle$. We use the periodic spline scaling function ϕ_k^j, $k = 0, \cdots, 2^j - 1$. We note $\theta_k^j = (I - \triangle)^{-1}\phi_k^j$ and recall (see section 1.8.4) that it can be approximated up to an error $\epsilon = 10^{-6}$ by an element in V_{j+3} [16]. To calculate the trace, we need to know the point values of these functions. It is completed by the algorithms described in section 1.2.1 and section 1.8.4. Quadrature formula are used to compute the matrices F_h. We use the third order formula:

$$\begin{aligned}\int_0^1 f(x)dx =& h[\frac{3}{8}f_1 + \frac{7}{6}f_2 + \frac{23}{24}f_3 + f_4 + f_5 + \\ & \cdots + f_{N-3} + f_{N-2} + \frac{23}{24}f_{N-1} + \frac{7}{6}f_N + \frac{3}{8}f_{N+1}] + O(h^4),\end{aligned} \tag{3.22}$$

where $0 = x_1 < x_2 < \cdots < x_N < x_{N+1} = 1$ is a regular segmentation of $[0, 1]$ with $h = \frac{1}{N}$, $x_i = (i-1)h$, $f_i = f(x_i)$, $i = 1, 2, \cdots, N+1$.

The last question we consider here is how to get the point value of u_h(for example at scale p, $p > j$) from its coefficients U_h at scale j. Note U_h^j for U_h at scale j, the reconstruction process provides that

$$U_h^j \to U_h^{j+1} \to \cdots \to U_h^p.$$

Then the interpolation process introduced in section 1.8.3 gives

$$u_h(\frac{k}{2^p}), k = 0, \cdots, 2^p - 1.$$

3.6.2 Cas bidimensionnels

Again $L = I - \triangle$ except for some special cases. $\phi_\alpha^\Omega(x, y) = \phi_{k_1}^j(x)\phi_{k_2}^j(y)$ where ϕ_k^j is the periodic spline scaling function. $\theta_\alpha^\Omega(x, y) = (I - \triangle)^{-1}\phi_\alpha^\Omega(x, y)$ approximated by an element in V_{j+3}, see section 1.8.4, $\alpha = (j, k_1, k_2)$, $0 \le k_1, k_2 \le 2^j - 1$. Each curve γ or Γ is supposed to be piecewise regular. $\phi_{\alpha'}^\Gamma$ and $\phi_{\alpha''}^\gamma$ are constructed by the method in section 1.3. When Γ is composed of lines, i.e. $\Gamma = \bigcup_{k\in K} \Gamma_k$ and two ends of Γ_k read (a_1^k, b_1^k) and (a_2^k, b_2^k). Then $C_{\alpha,\alpha'}$ is the combination of terms as follows:

$$C_{\alpha,\alpha'}^k = \sqrt{(a_1^k - a_2^k)^2 + (b_1^k - b_2^k)^2} \int_0^1 T_\Gamma \theta_\alpha^\Omega((1-\tau)a_1^k + \tau a_2^k, (1-\tau)b_1^k + \tau b_2^k)\phi_{\alpha'}^{[0,1]}(\tau)d\tau.$$

where $\alpha' = (j', l_1')$, $l_1' = 0, \cdots, 2^{j'} - 1$. This integration is obtained from the point value of function $T_\Gamma \theta_\alpha^\Omega((1-\tau)a_1^k + \tau a_2^k, (1-\tau)b_1^k + \tau b_2^k)$ at scale j' by the relation between the point value of function and its coefficient with scaling function. Better precision of calculation of this integration can be achieved by upgrading j' to $j' + r$, $r > 0$ and applying the decomposition process. The same method can be applied to the calculation of $D_{\alpha'',\alpha}$ and $(G_h)_{\alpha''}$. For $(F_h)_\alpha$, we use formula with $\Omega = [0,1] \times [0,1]$

$$\int_\Omega f(x)dx = h^2 \sum_{i=1}^{N+1} \sum_{j=1}^{N+1} C(i)C(j) f_{i,j},$$

where $0 = x_1 < x_1 < \cdots < x_N < x_{N+1} = 1$, $h = \frac{1}{N}$, $x_i = (i-1)h$, $i = 1, \cdots, N+1$, $N = 2^{j+3}$, $f_{i,j} = f(x_i, x_j)$. The coefficient $C(\cdot)$ is defined as following

$$C(i) = \begin{cases} 3/8, i = 1, N+1, \\ 7/6, i = 2, N, \\ 23/24, i = 3, N-1, \\ 1, (other). \end{cases}$$

This is an extension of the quadrature formula we have used in 1 dimension.

3.7 Résultats numériques

3.7.1 Cas unidimensionnels

Convergence of the derivative jump

The initial problem is

$$\begin{cases} u - \triangle u = f, \text{ on }]0, 0.5[\\ u(0) = 0, u(0.5) = 0, \end{cases}$$

with $f(x) = (1 + (2\pi)^2)sin(2\pi x)$. We choose $a' = a = 0$, $b = b' = 0.5$ and $\tilde{f}$, the extension of f is defined as follows:

$$\tilde{f}(x) = \begin{cases} f(x) & \text{on }]a, b[, \\ -(1 + (2\pi)^2)sin(2\pi x) & \text{on } [0, 1]\setminus]a, b[. \end{cases} \tag{3.23}$$

Here the spline order is $m = 4$. The exact derivative jump at $b' = b = 0.5$ is 12.5664. The numerical estimates for different value of j are given in table 3.1.

Table 3.1: Numerical derivative jump at $b' = b$ versus scale j

j	λ_2
6	12.5739
7	12.5702
8	12.5683
9	12.5673

Global error in $L^2]a, b[$

We consider the same problem with $a = 0.2$, $b = 0.5$. We choose $a' = 0.1$, $b' = 0.7$ and smoothly extend f as $\tilde{f}(x) = (1 + (2\pi)^2)sin(2\pi x)$. Again the spline order is $m = 4$. Under this extension, we know that the derivative jump at the boundary $\lambda = 0$. The numerical estimates of $\|u - u_j\|_{L^2]a,b[}$ agrees with the theoretical estimation with a slope $-mlog2$ in logarithmic scale, see figure 3.2. If one takes again $\tilde{f}(x) = -(1 + (2\pi)^2)sin(2\pi x)$ on $[0, 1]\setminus]a, b[(\lambda \neq 0)$, then we expect the numerical error in logarithmic scale to decrease at a rate $-\frac{3}{2}log2$. The results of the smooth and old fictitious methods are compared in table 3.2 and figure 3.3. It is seen that the convergence rate with the smooth fictitious domain method is not limited by $-\frac{3}{2}log2$.

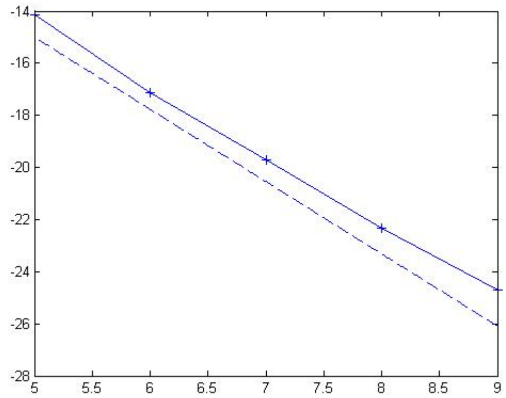


Figure 3.2: +: numerical error in $L^2[a,b]$ in semi-logarithmic scale at $j = 5, 6, 7, 8, 9$ with $\lambda = 0$, Solid curve: line connecting the points +, dashed curve: line of slope $-4log2$

Table 3.2: Error in $L^2[a,b]$ of old and smooth method with a non smooth $\tilde{f}$

j	old method	smooth method
4	2.517012e-002	4.044385e-003
5	1.246856e-002	2.607095e-004
6	6.155570e-003	5.880814e-005
7	3.051376e-003	1.342806e-005

Interior error

Here $L = (I - \nu\triangle)$, $\nu = \frac{1}{64\pi^2}$, $a = 0.2$, $b = 0.7$, the exact solution is $u = x^5 - x^3 + 12x^2 - 2.5x + 2$ and $f = Lu$ on $]a,b[$. We choose $a' = 0.1$, $b' = 0.9$, $m = 6$ and extension of $\tilde{f}$ constant on $[0,a](\tilde{f} = f(a))$ and on $[b,1](\tilde{f} = f(b))$. The interior error in $L^2[0.4, 0.5]$, $H^1[0.4, 0.5]$ are evaluated in semi-logarithmic scale in figure 3.4 using the smooth method. From theorem 3.22, it follows that the error in norm $L^2[0.4, 0.5]$(resp $H^1[0.4, 0.5]$) decreases at rate $-4log2(-3log2)$.

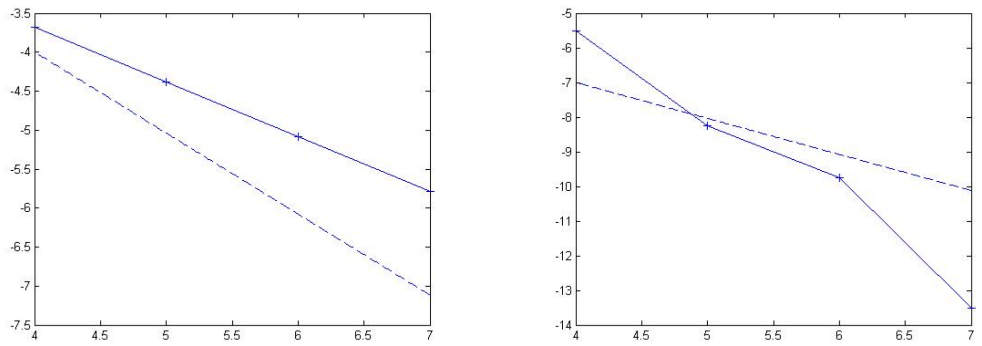


Figure 3.3: Numerical error in $L^2[a,b]$ in semi-logarithmic scale versus j(dashed line: slope $-\frac{3}{2}log2$), $\lambda \neq 0$. Left old method, right smooth method.

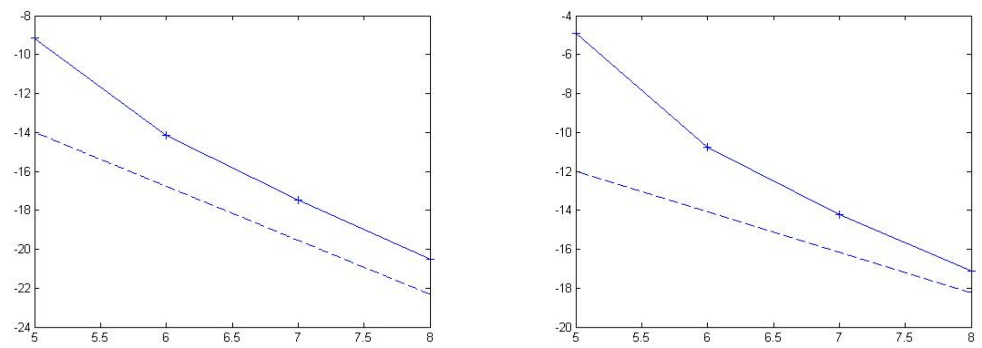


Figure 3.4: Smooth method. Left figure: in norm $L^2[0.4, 0.5]$, dashed curve with slope $-4log2$, right figure: in norm $H^1[0.4, 0.5]$, dashed curve with slope $-3log2$

Comparison between the smooth and the old fictitious method

The errors in $L^2[a,b]$ for different values of j are listed in table 3.3 and plotted on figure 3.5 in semi-logarithmic scale. Figure 3.6 shows the numerical solution for

Table 3.3: Error in $L^2[a,b]$ old and smooth method with constant extension for f

j	old method	smooth method
4	1.924962e-002	3.374893e-003
5	1.224160e-002	1.236899e-004
6	5.201504e-003	6.146370e-006
7	2.614771e-003	5.550808e-008

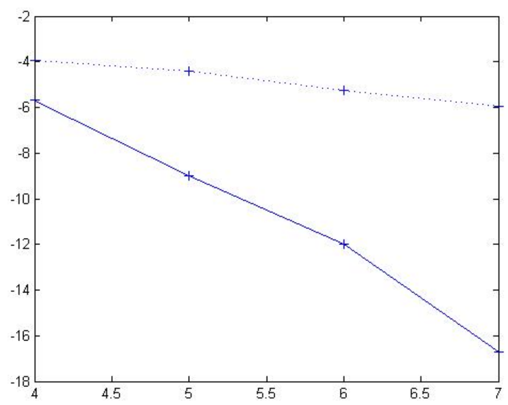


Figure 3.5: +: numerical error in $L^2[a,b]$ in semi-logarithmic scale at $j = 4, 5, 6, 7$, Solid curve: smooth method, dotted curve: old method

j=7. We clearly see from this figure that the smooth fictitious domain smoothly extend the numerical solution from the original domain $]0.2, 0.5[$ to $]0.1, 0.7[$. The derivative jump is on control points, i.e. 0.1 or 0.7.

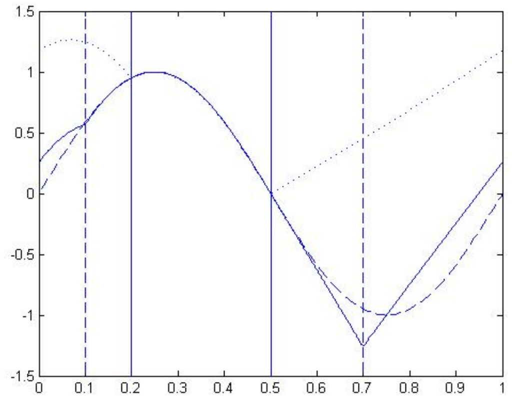


Figure 3.6: The dashed curve is exact solution $sin(2\pi x)$, dotted one is for the old method and solid one is for the smooth method at scale $j = 7$ with constant extension of f

Evaluation of the condition $(DC)^t(DC)$

We take $a = a' = 0$, $b = 0.2$, $b' = 0.5$, a plot for different values of j the condition number of $(DC)^t(DC)$ on table 3.4. This estimate confirms that it is independent of scale j. Table 3.5 exhibits dependance of the condition number on $|b' - a'|$ for $a =$

Table 3.4: Condition number of $(DC)^t(DC)$ with scale j

j	$cond_2(DC)^t(DC)$
5	1.8494e+003
6	1.8042e+003
7	1.7823e+003
8	1.7713e+003

$a' = 0$, $b = 0.2$, $j = 7$. For the condition number of C^tC, the numerical result agrees with the theoretical estimation. Namely, it decreases with the increment of $|b' - a'|$. However, this trend is not followed by the condition number of $\text{cond}_2(DC)^tDC$.

Table 3.5: Condition number with b'

b'	$cond_2(DC)^t(DC)$	$cond_2C^tC$
0.2	7.0397e+002	1.9643e+003
0.3	8.9818e+002	1.1419e+003
0.4	1.2185e+003	8.7528e+002
0.5	1.7713e+003	8.0698e+002

.

3.7.2 Cas bidimensionnels

Observation of the decrease rate of the error in $L^2(\omega)$

Here $L = I - \triangle$, $f = (1+(2\pi)^2)(sin(2\pi x) + cos(2\pi y))$ in $\Omega = [0,1] \times [0,1]$ and the boundary condition of f on γ reads $g = sin(2\pi x) + cos(2\pi y)$, $\omega = [0,1] \times [0,0.5]$, $\Xi = [0,1] \times [0,0.7]$. The family of basis on the boundary γ and Γ are constructed from the compactly support wavelet on $[0,1]$ with $N = 4$ vanishing moments. Here $j' = j'' = 4$ and the spline order is $m = 4$. As in the univariate case, we recover a decay rate of $-mlog(2)$ when $\lambda = 0$(smooth extension)(figure 3.7) and of $-\frac{3}{2}log2$ when $\lambda \neq 0$ ($\tilde{f} = (1+(6\pi)^2+(2\pi)^2)sin(6\pi x)sin(2\pi y)$ in ω, $\tilde{f} = -(1+(6\pi)^2+(2\pi)^2)sin(6\pi x)sin(2\pi y)$ in $\Omega \setminus \omega$, $g = sin(6\pi x)sin(2\pi y)$, $N = 2$, $j' = j'' = 3$)(figure 3.8).

Comparison between the smooth and the old fictitious method

Here we adopt the example with $g = sin(2\pi x)cos(4\pi y)$, spline order 6, $j = 5$, $j' = j'' = 5$, $N = 4$, $\omega = [0,1] \times [0.2, 0.5]$, $\Xi = [0,1] \times [0.1, 0.7]$. Function f is constantly extended along the y direction

$$f(x,y) = \begin{cases} (1+(2\pi)^2+(4\pi)^2)sin(2\pi x)cos(4\pi \times 0.2), & y < 0.2, \\ (1+(2\pi)^2+(4\pi)^2)sin(2\pi x)cos(4\pi \times 0.5), & y > 0.5, \\ (1+(2\pi)^2+(4\pi)^2)sin(2\pi x)cos(4\pi y), & else. \end{cases}$$

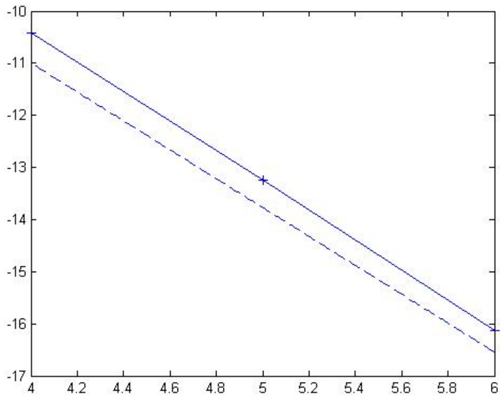


Figure 3.7: +: error in $L^2(\omega)$ in semi-logarithmic scale at scale $j = 4, 5, 6$, solid line: curve connecting +, dashed line: curve of slope $-4log2$ with $\lambda = 0$

The error in norm $L^2(\omega)$ is 3.448604e-003 for the smooth method in comparison to 2.724515e-002 for the old one. The plot of the different solutions can be seen on figure 3.9. The cross-sections along $x = 0.25$ and $x = 0.625$ are visible on figure 3.10.

Another example is presented with spline order 6, $j' = j'' = 6$, $N = 4$, $g = 2cos(4\pi x)cos(2\pi y)$, $\omega = [0.375, 0.625] \times [0.375, 0.625]$, $\Xi = [0.2, 0.8] \times [0.2, 0.8]$ for the smooth fictitious method while $\Xi = [0.375, 0.625] \times [0.375, 0.625]$ for the old one. Function f is extended to the whole domain Ω as

$$f(x,y) = \begin{cases} 2(1+(2\pi)^2+(4\pi)^2)cos(4\pi x)cos(2\pi y), & (x,y) \in [0.375, 0.625] \times [0.375, 0.625], \\ 2(1+(2\pi)^2+(4\pi)^2)cos(4\pi \times 0.375)cos(2\pi y), & x < 0.375, \\ 2(1+(2\pi)^2+(4\pi)^2)cos(4\pi \times 0.625)cos(2\pi y), & x > 0.625, \\ 2(1+(2\pi)^2+(4\pi)^2)cos(4\pi x)cos(2\pi \times 0.375), & (x,y) \in [0.375, 0.625] \times [0, 0.375], \\ 2(1+(2\pi)^2+(4\pi)^2)cos(4\pi x)cos(2\pi \times 0.625), & (x,y) \in [0.375, 0.625] \times [0.625, 1]. \end{cases}$$

The evolution of the errors $L^2(\omega)$ versus the scale j for the smooth and old method are presented in table 3.6 and figure 3.11 in semi-logarithmic scale. The plots of the solutions at scale $j = 7$ are seen on figure 3.12. The cross-sections along $x = 0.5$ and $y = 0.4375$ are on figure 3.13.

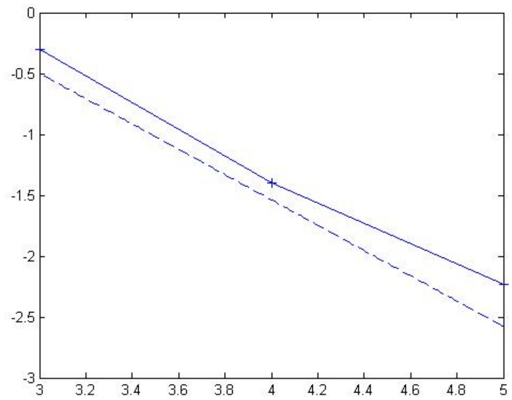


Figure 3.8: +: error in $L^2(\omega)$ in semi-logarithmic scale versus j, dashed line: theoretical estimate, slope $-\frac{3}{2}log2$ ($\lambda \neq 0$)

Table 3.6: Error in $L^2(\omega)$ for the smooth and old fictitious methods versus the scale j

j	old method	smooth method
5	1.570196e-002	2.001687e-003
6	7.726092e-003	1.234283e-004
7	6.814569e-003	1.847159e-006

Interior error estimate

We consider the problem $(2I-3\triangle)u = f$, $u = sin(\pi x)+sin(\pi y)+cos(\pi x)+cos(\pi y)+x^6+y^6$, $\omega = [0.35, 0.62] \times [0.35, 0.62]$, $\Xi = [0.2, 0.7] \times [0.2, 0.7]$, f is constantly extended along the outer normal direction. The parameter are $j' = j'' = 6$, $N = 4$, $m = 6$, $\omega_0 = [0.4, 0.5] \times [0.4, 0.5]$. The norms $L^2(\omega_0)$ and $H^1(\omega_0)$ of the errors are plotted versus the scale j on Figure 3.14. From theorem 3.23, it follows that the error in norm $H^1(\omega_0)$ decreases at rate of $-5log2$.

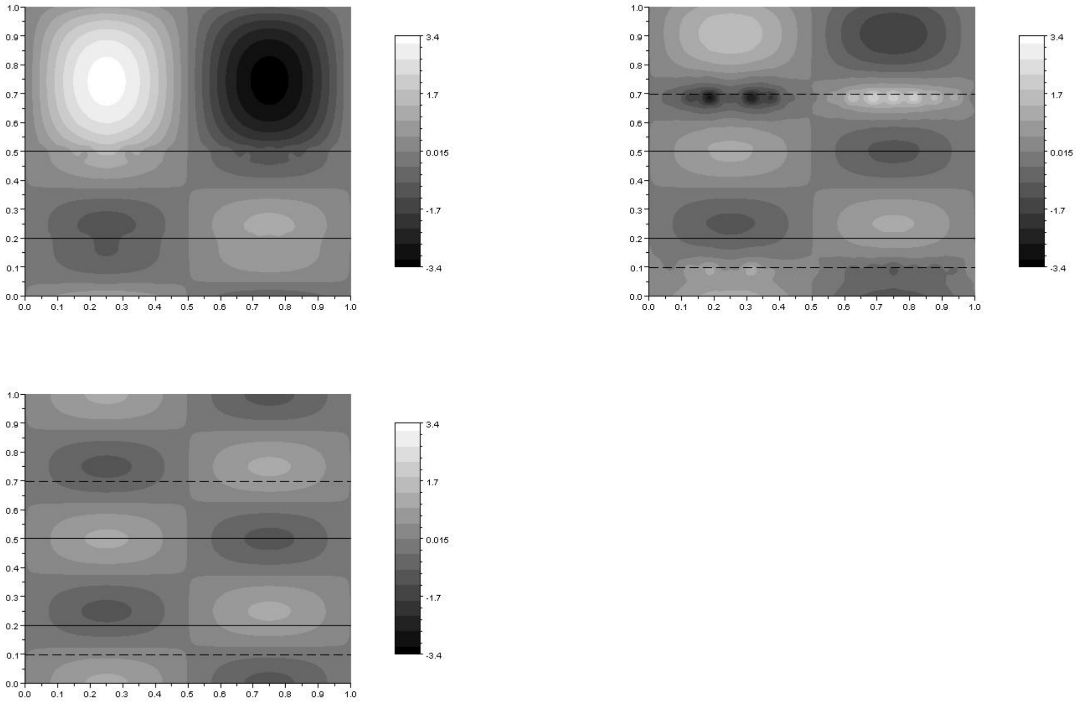
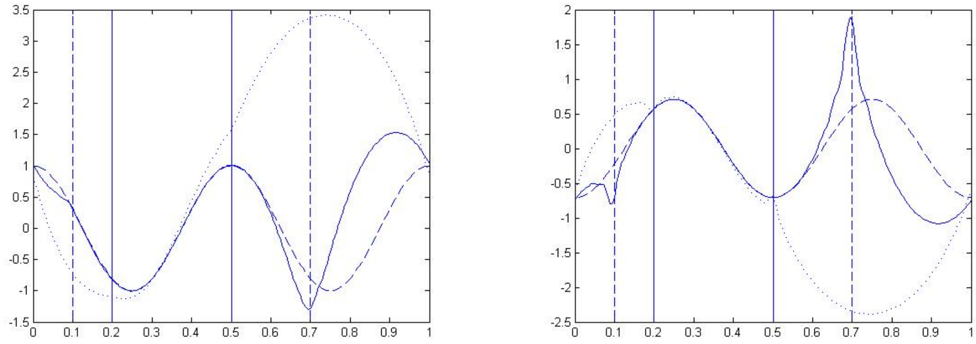


Figure 3.9: top: left figure with old method while the right one with smooth method, $\omega = [0,1] \times [0.2, 0.5]$, $\Xi = [0,1] \times [0.1, 0.7]$, bottom: exact solution. line: real boundary γ, dashed line :control boundary Γ.

Evaluation of the condition number of $(DC)^t(DC)$

We display the evolution of the condition number versus the parameters $j, j'(j' = j'')$ to check the theoretical estimation. The table 3.7 shows the condition number versus j$(j' = j'' = 4)$. The dependance on j is weaker than our theoretical estimation. The influence of $j'(j' = j'')(j = 6)$3.15 on condition number is in agreement with our prediction with slope $log4$ in semi-logarithmic scale.



(a) cross-section along $x = 0.25$ (b) cross-section along $x = 0.625$

Figure 3.10: smooth method(solid line); old method(dotted line); exact solution(dashed line) at $j = 5$

Table 3.7: Condition number of $(DC)^t(DC)$ with scale j

j	$cond_2(DC)^t(DC)$
5	1.6134e+009
6	2.0011e+009
7	1.7781e+009

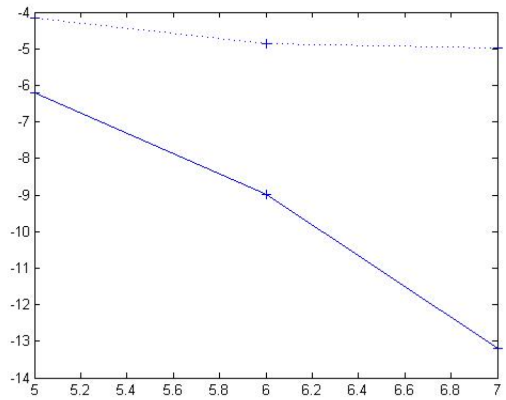


Figure 3.11: +: numerical error in $L^2(\omega)$ in semi-logarithmic scale at $j = 5, 6, 7$, Solid curve: smooth method, dotted curve: old method

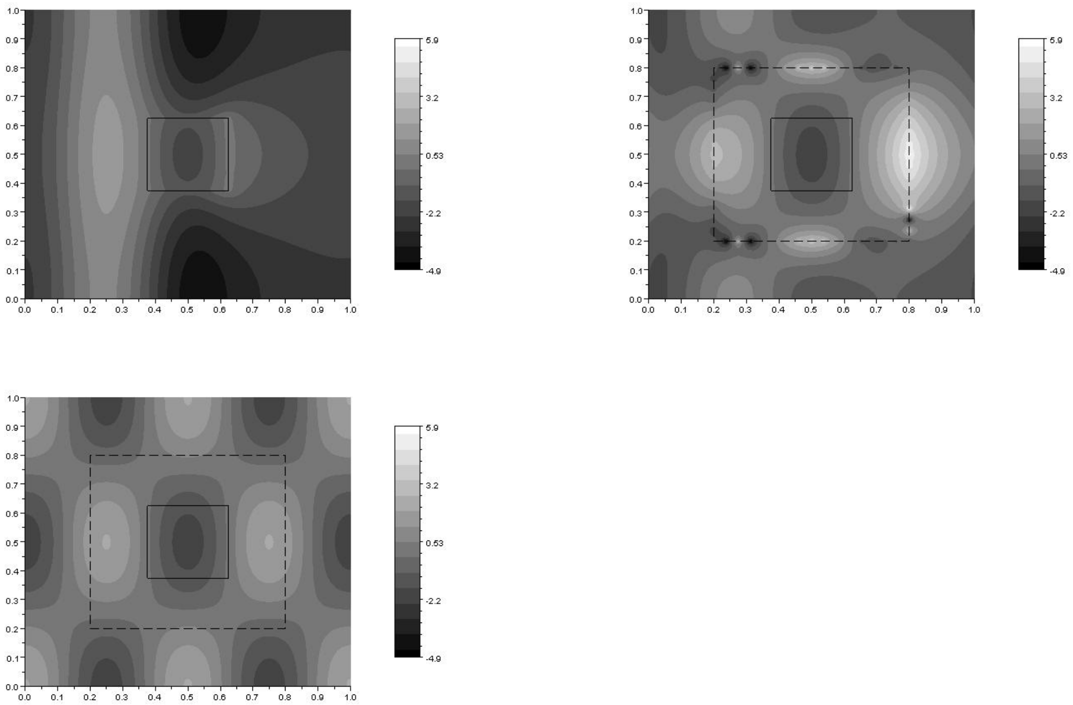
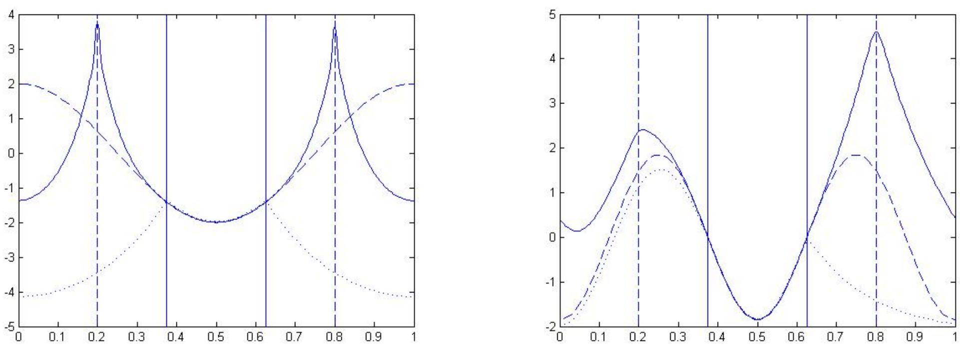


Figure 3.12: top: left figure with old method while the right one with smooth method, $\omega = [0.375, 0.625] \times [0.375, 0.625]$, $\Xi = [0.2, 0.8] \times [0.2, 0.8]$, bottom: exact solution. line: real boundary γ, dashed line :control boundary Γ.



(a) cross-section along $x = 0.5$ (b) cross-section along $y = 0.4375$

Figure 3.13: solid curve: smooth method, dotted curve: old method, dashed curve: exact solution at scale $j = 7$

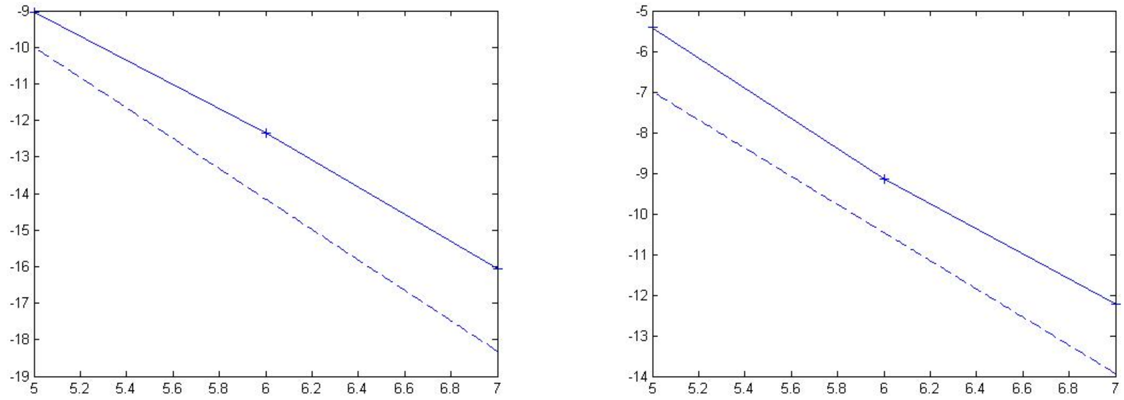


Figure 3.14: left: error in norm $L^2(\omega_0)$, dashed line for theoretical estimate of slope -6log(2); right: error in norm $H^1(\omega_0)$, dashed line for theoretical estimate of slope -5log(2)

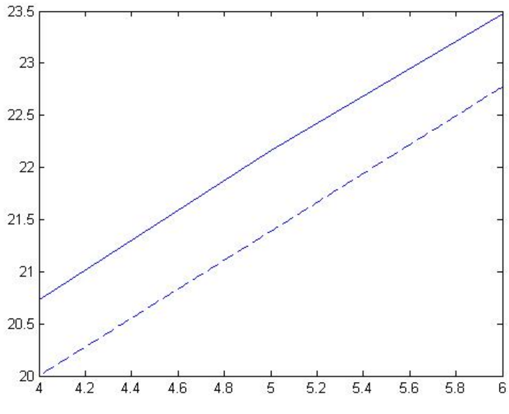


Figure 3.15: Evolution of condition number with $j'(j' = j'')$ in semi-logarithmic scale,dashed line: theoretical result of slope log(4), solid line: numerical result

Chapter 4

Prolongement lisse utilisant la construction d'ondelettes à support compact sur l'intervalle

Ce chapitre introduit une technique de prolongement lisse utilisant les analyes multirésolutions sur l'intervalle. Il existe deux situations de prolongement lisse: soit f est connue sur tout les points de ω, soit f est connue sur une grille fixe dans ω. Nous exposons les techniques de prolongement en dimension 1 où $\omega = [a, b]$. Pour la première situation, nous utilisons la multirésolution sur l'intervalle proposée par A. Cohen ou l'échelle j est choisi $j \geq j_0 = log_2(2N)$(N étant le nombre de moment nul). Cette construction définit les N fonctions de bord de chaque côté et les $2^j - 2N$ fonctions intérieures. Nous montrons que toute fonction de bord peut être étendue avec la régularité $C^{0.2N}$ en enlevant la restriction sur l'interval $[0, 1]$. En conséquence, l'approximation de f sur $[a, b]$ peut être prolongé sur $I\!R$ avec régularité $C^{0.2N}$. Pour la seconde situation, nous utilisons la multirésolution sur l'intervalle à grande échelle proposée par S. Bertoluzza. Une base de Riesz est construite pour cette multirésolution. Comme dans le cas précédent, le fait d'enlever la restriction à $[0, 1]$ effectue un prolongement sur $I\!R$ de régularité $C^{0.2N}$. Ces deux méthodes de prolongement lisse sont généralisées à un convexe en dimension 2. Nous donnons des exemples de prolongements lisses et appliquons les techniques à la méthode de domaine fictif. Ces techniques de prolongement lisse permettent en particulier de vérifier les estimations intérieures du chapitre précédant.

When applying the smooth fictitious domain method, it appeared that the smoothness of the extended source term f plays a key role in the analysis of error decrease rate. In the previous examples, we had an explicit expression for f on ω that was moreover tractable on $\Omega \setminus \omega$. In practice, the situation may be less easy; either the expression for f can be meaningless outside ω, or there does not exist any expression for f and one must perform an extension using the available evaluation of f on grid points in the interior of ω. The problem of the extension outside a domain ω have already been addressed, in a general context (see for instance [30] and references in it) or in the context of fictitious domain with Lagrange multipliers ([52]). Different extension techniques are introduced by M.Elghaoui [30] and M.Mommer [52]. In particular, M.Mommer construct a method which produces optimally smooth extension of the solution. It proposes to formulate on the fictitious domain a rank-deficient, but otherwise well-posed, least squares problem whose solutions all agree on the original domain with the original solution. The smoothness is recovered by a discrete operator which is proven to work under certain additional condition. However there still exist the following problems: the inefficiency of discrete operator(large number of iterations needed), theoretical gap for discrete operator(in respect of convergence and smoothness) and the lack of adaptability of this method to problems on domains that contain corners. Here we introduce a new smooth extension method, which uses the regularity of orthonormal compact support scaling function proposed by I. Daubechies[26] and the construction of wavelets for the interval [21].

4.1 Procédure de prolongement, pour une fonction connue en tous points de ω

4.1.1 Introduction

We consider the situation where the function f we want to extend can be evaluated on any point of ω. We revisit the content of section 1.2.2 devoted to the construction of a multiresolution of $L^2(]0,1[)$ and focus on the construction of the basis of scaling functions of V_j. To construct the 2^j elements of the basis, we first consider the set $\{\phi_{j,k}, k = N, \cdots, 2^j - N - 1\}$, called the interior functions. Their support are all included in $[0,1]$. In addition $2N$ functions are required to complete the whole basis of V_j. At each side, N edge functions are added, they are selected to ensure the polynomial exactness of order $N-1$. This means that together with interior functions they generate all polynomials on $[0,1]$ up to degree $N-1$. Focusing on the left edge, we define

$$\tilde{\phi}_k^0(x) = \sum_{n=k}^{2N-2} \binom{n}{k} \phi(x+n-N+1)\chi_{[0,+\infty)}, \quad k = 0, \cdots, N-1,$$

with $\chi_{[0,+\infty)}$ the characteristic function on interval $[0,+\infty)$. The supports of these functions are staggered, i.e. support $\tilde{\phi}_k^0(x) = [0, 2N-1-k]$. Their properties are described as follows[21],

Lemma 4.1. *The N functions $\tilde{\phi}_k^0$, $k = 0, \cdots, N-1$ are independent, and orthogonal to the $\phi_{0,m}$, $m \geq N$. With the $\phi_{0,m}$, $m \geq N$, they generate all the polynomials up to degree $N-1$ on $[0,\infty)$. Finally, there exist constants $a_{k,l}$, $b_{k,m}$ such that*

$$\tilde{\phi}_k^0 = \sum_{l=0}^{k} a_{k,l}\tilde{\phi}_l^0(2x) + \sum_{m=N}^{3N-2-2k} b_{k,m}\phi(2x-m). \tag{4.1}$$

The left edge scaling functions $(\phi_k^0)_k$ are obtained by reordering and orthonormalizing $(\tilde{\phi}_k^0)_k$. They still have staggered supports since support $\phi_k^0 = [0, N+k]$. There exists a matrix T^0 such that

$$\begin{pmatrix} \phi_0^0 \\ \phi_1^0 \\ \vdots \\ \phi_{N-1}^0 \end{pmatrix} = T^0 \begin{pmatrix} \tilde{\phi}_0^0 \\ \tilde{\phi}_1^0 \\ \vdots \\ \tilde{\phi}_{N-1}^0 \end{pmatrix}.$$

Remark 4.1.1. We orthonormalize $\{\tilde{\phi}_k^0, k = 0, \cdots, N-1\}$ by a Gramm-Schmidt procedure, starting with $\tilde{\phi}_{N-1}^0$ to lower values of k, $\tilde{\phi}_k^0$, to get $\{\phi_k^0, k = 0, \cdots, N-1\}$.

We denote by $\phi^0_{j,k}(x) = 2^{\frac{j}{2}}\phi^0_k(2^j x)$. They satisfy a modified scaling equation

$$\phi^0_{j,k} = \sum_{l=0}^{N-1} H^0_{k,l}\phi^0_{j+1,l} + \sum_{l=N}^{N+2k} H^0_{k,l}\phi_{j+1,l}, \quad k = 0, \cdots, N-1.$$

The right edge scaling functions $\phi^1_{j,k}$ are constructed as follows. Let the filter coefficients $h^\sharp_n = h_{-n+1}$, then $\phi^\sharp(x) = \phi(1-x)$ and note

$$\tilde{\phi}^{0,\sharp}_k(x) = \sum_{n=k}^{2N-2} \binom{n}{k} \phi^\sharp(x+n-N+1)\chi_{[0,+\infty)}, \quad k = 0, \cdots, N-1.$$

They lead to the new left edge scaling function $\phi^{0,\sharp}_{j,k}$,

$$\phi^{0,\sharp}_{j,k} = \sum_{l=0}^{N-1} H^{0,\sharp}_{k,l}\phi^{0,\sharp}_{j+1,l} + \sum_{l=N}^{N+2k} H^{0,\sharp}_{k,l}\phi^\sharp_{j+1,l}, \quad k = 0, \cdots, N-1,$$

and there exists a matrix T^1 such that

$$\begin{pmatrix} \phi^{0,\sharp}_0 \\ \phi^{0,\sharp}_1 \\ \vdots \\ \phi^{0,\sharp}_{N-1} \end{pmatrix} = T^1 \begin{pmatrix} \tilde{\phi}^{0,\sharp}_0 \\ \tilde{\phi}^{0,\sharp}_1 \\ \vdots \\ \tilde{\phi}^{0,\sharp}_{N-1} \end{pmatrix}.$$

Define

$$\phi^1_{j,k}(x) = \phi^{0,\sharp}_{j,k}(1-x), \quad k = 0, \cdots, N-1.$$

Together with the interior scaling function $\phi_{j,k}$, $k = N, \cdots, 2^j - N - 1$, they form an orthonormal basis of $V_j[0,1]$. For $\forall f_j \in V_j[0,1]$,

$$f_j(x) = \sum_{k=0}^{N-1} C^0_{j,k}\phi^0_{j,k} + \sum_{k=N}^{2^j-N-1} C_{j,k}\phi_{j,k} + \sum_{k=0}^{N-1} C^1_{j,k}\phi^1_{j,k}.$$

Remark 4.1.2. The supports of left edge scaling function, interior scaling function, right edge scaling function are $[0, 2^{-j}(2N-1)]$, $[2^{-j}, 1-2^{-j}]$, $[1-2^{-j}(2N-1), 1]$ respectively. So the coarsest level J of j is chosen $2^J \geq 2N$ such that the left and right edge function do not have overlapping support. To make smooth extension of an function, we concentrate on how to smoothly extend the edge scaling function.

We note that the function $\phi^0_{j,k}$, $\phi^1_{j,k}$ have the same regularity as ϕ given by $\phi \in C^{0.2N}$.

4.1.2 De l'intervalle [0,1] à $I\!R$

Taking the left edge function for example, we obtain a natural extension on $]-\infty,1[$ with regularity $C^{0.2N}$ removing the restriction to $[0,1]$ in $\tilde{\phi}^0_{j,k}$. Then introducing

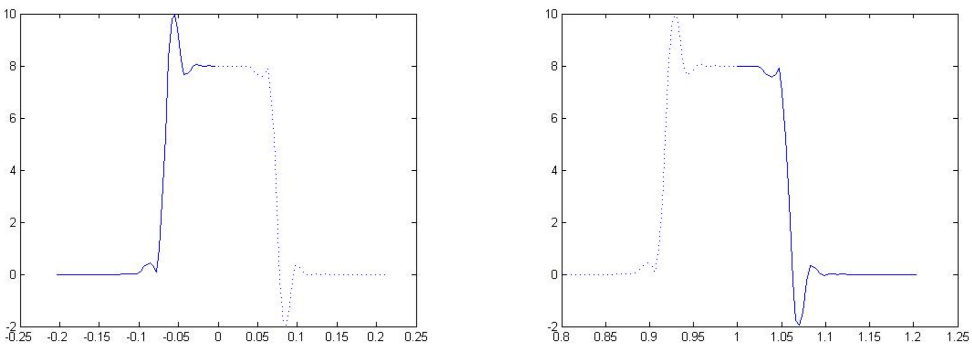
$$\hat{\phi}^0_{j,k}(x) = 2^{\frac{j}{2}} \sum_{n=k}^{2N-2} \binom{n}{k} \phi(2^j x + n - N + 1), \quad k = 0, \cdots, N-1,$$

the support of $\hat{\phi}^0_{j,k}$ is $[-\frac{2N-2}{2^j}, \frac{2N-1-k}{2^j}]$, $k = 0, \cdots, N-1$. As the support of interior basis is completely included in $[0,1]$, we extend them by 0 at the left and right side.

The according functions at the right edge are

$$\hat{\phi}^1_{j,k}(x) = 2^{\frac{j}{2}} \sum_{n=k}^{2N-2} \binom{n}{k} \phi^\sharp(2^j(1-x) + n - N + 1), \quad k = 0, \cdots, N-1.$$

Their supports $\hat{\phi}^1_{j,k}(x) = [1 - \frac{2N-1-k}{2^j}, 1 + \frac{2N-2}{2^j}]$. So for every $f \in V_j[0,1]$, a smooth extension of f on the whole line(with compact support $[-\frac{2N-2}{2^j}, 1 + \frac{2N-2}{2^j}]$) can be obtained. Then we see an extension example with $N = 5$(figure 4.1), the extension part denoted by solid line.



(a) $\hat{\phi}^0_{6,0}$ (b) $\hat{\phi}^1_{6,0}$

Figure 4.1: smooth extension for the edge function

Finally, we construct an element of $V_j(I\!R)$. Its restriction on $[0,1]$ coincides with the interpolation by wavelets on the interval $[0,1]$.

4.1.3 D'un intervalle arbitraire [a,b] à $I\!R$

We introduce a multiresolution analysis on $L^2[a,b]$ using the linear mapping

$$B^{-1}\begin{cases} I\!R \mapsto I\!R, \\ x \mapsto \dfrac{x}{b-a} + \dfrac{a}{a-b}. \end{cases}$$

Like on the interval $[0,1]$, we construct the nested spaces $V_j[a,b]$ of dimension 2^j, $2^J \geq 2N$

$$V_J[a,b] \subset V_{J+1}[a,b] \subset \cdots \subset \cdots \subset \mathrm{L}^2[a,b],$$

$$\overline{\bigcup_{j\geq J} V_j[a,b]} = L^2[a,b].$$

$V_j[a,b]$ is generated by the orthogonal family

$$V_j[a,b] = span \left\{ \begin{array}{ll} \phi^0_{j,k}(B^{-1}x), & k = 0,\cdots,N-1, \\ \phi_{j,k}(B^{-1}x), & k = N,\cdots,2^j-N-1, \\ \phi^1_{j,k}(B^{-1}x), & k = 0,\cdots,N-1. \end{array} \right\}.$$

Remark 4.1.3. The supports of left edge scaling function, interior scaling function, right edge scaling function are $[a, a+2^{-j}(2N-1)(b-a)]$, $[a+2^{-j}(b-a), a+(b-a)(1-2^{-j})]$ and $[a+(b-a)(1-2^{-j}(2N-1)), b]$, respectively. The new family basis are orthonormal with the inner product $(f,g) = \int \frac{1}{b-a} fg dx$, $\forall f,g \in L^2[a,b]$.

We write

$$\tilde{\phi}^0_{j,k}(x) = 2^{\frac{j}{2}} \sum_{n=k}^{2N-2} \binom{n}{k} \phi(2^j B^{-1}x + n - N + 1)\chi_{[a,b]}, \quad k = 0,\cdots,N-1,$$

$$\hat{\phi}^0_{j,k}(x) = 2^{\frac{j}{2}} \sum_{n=k}^{2N-2} \binom{n}{k} \phi(2^j B^{-1}x + n - N + 1), \quad k = 0,\cdots,N-1.$$

Their supports are $\hat{\phi}^0_{j,k} = [a - \frac{(2N-2)(b-a)}{2^j}, a + \frac{(2N-1-k)(b-a)}{2^j}]$.

The according functions at the right edge read

$$\tilde{\phi}^1_{j,k}(x) = 2^{\frac{j}{2}} \sum_{n=k}^{2N-2} \binom{n}{k} \phi^\sharp(2^j(1 - B^{-1}x) + n - N + 1)\chi_{[a,b]}, \quad k = 0,\cdots,N-1,$$

$$\hat{\phi}^1_{j,k}(x) = 2^{\frac{j}{2}} \sum_{n=k}^{2N-2} \binom{n}{k} \phi^\sharp(2^j(1 - B^{-1}x) + n - N + 1), \quad k = 0, \cdots, N-1.$$

Their supports $\hat{\phi}^1_{j,k}(x) = [a + (b-a)(1 - \frac{2N-1-k}{2^j}), a + (b-a)(1 + \frac{2N-2}{2^j})]$. So for every $f \in L^2[a,b]$, we approximate it with $f_j \in V_j[a,b]$, then smoothly extend f_j on the whole line on support $[a - \frac{(2N-2)(b-a)}{2^j}, a + (b-a)(1 + \frac{2N-2}{2^j})]$.

4.1.4 D'un convexe $\omega \subset I\!R^2$ à $I\!R^2$

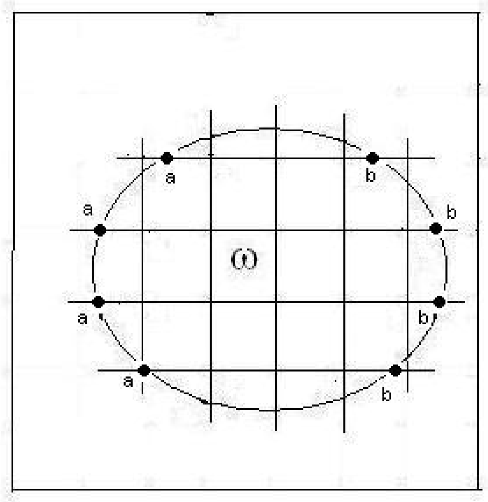
We first do the extension in x direction, taking the cross points as a and b. Then the smooth extension process applies in y direction and divides into two cases: the line in y direction has cross points with ω or not. The former we use the cross points as a and b. The latter we take the y coordinate of the lowest and highest line in x direction, which have cross points with ω, as a and b. This procedure see on figure 4.2.

4.1.5 Implémentation

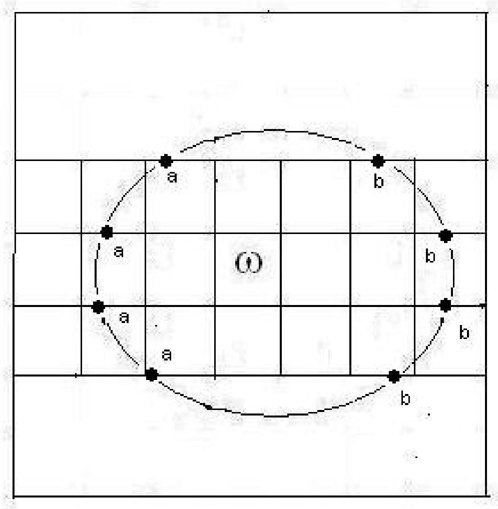
To estimate the coefficients with edge scaling functions, we use a quadrature formula proposed in [15]. This formula is of order $N-1$. The key of the method is the computation of the N moments of every edge function. Indeed if one looks for the weight coefficient $w_{i,k}$ such that

$$\int f\phi^0_{0,k} \approx \sum_{i=0}^{N-1} w_{i,k} f(a_i), \quad k = 0, \cdots, N-1,$$

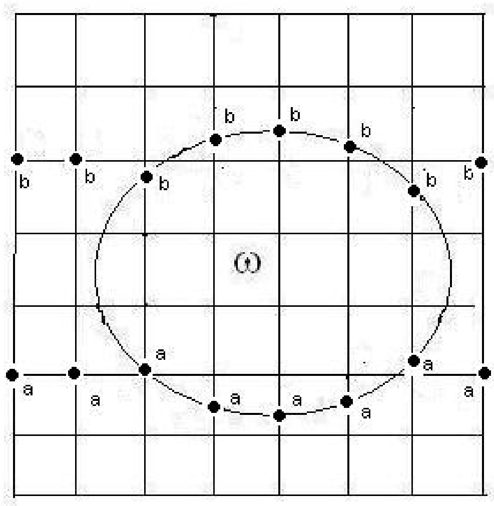
where $a_i \in$ support $\phi^0_{0,k}$. At scale $j = 0$, we set $a_i = i$, $i = 0, \cdots, N-1$. Since that this formula is exact for polynomial of order up to $N-1$, one needs to know the first N moments of every edge scaling function. This can be done by multiplying the modified scaling equation by x^l, $l = 0, \cdots, N-1$ and resolving the deduced linear systems. In this process, we calculate the first N moments of interior scaling



(a)



(b)



(c)

Figure 4.2: extension procedure

function by the following recurrence relation [19],

$$\begin{cases} M_k = \displaystyle\int x^k \phi(x)dx, \\ M_0 = 1, \\ M_k(1 - 1/2^k) = 1/2^{1/2+k} \displaystyle\sum_{n=-N+1}^{N} h_n \sum_{l=0}^{k-1} \binom{k}{l} n^{k-l} M_l. \end{cases}$$

The filter coefficients h_n are cited in [27] and $\sum_n h_n = \sqrt{2}$. When $N = 5$, $M_1 = -1.163430e{-}001$, $M_2 = 1.353569e{-}002$, $M_3 = -5.375272e{-}001$, $M_4 = 2.496004e{-}001$.

On the interval $[a, a + \frac{b-a}{2^j}]$, $f \approx f_j = \sum_{k=0}^{N-1} C^0_{j,k} \phi^0_{j,k}(B^{-1}(x))$,

$$C^0_{j,k} = 2^{-j/2} \int f(B(2^{-j}x)) \phi^0_{0,k}(x)dx,$$

with map B:

$$B \begin{cases} I\!R \mapsto I\!R, \\ x \mapsto a + (b-a)x. \end{cases}$$

It can be extended on the left edge as follows,

$$f_j = \begin{pmatrix} C^0_{j,0} & C^0_{j,1} & \cdots & C^0_{j,N-1} \end{pmatrix} T^0 \begin{pmatrix} \tilde{\phi}^0_{j,0}(x) \\ \tilde{\phi}^0_{j,1}(x) \\ \vdots \\ \tilde{\phi}^0_{j,N-1}(x) \end{pmatrix}$$

$$\xrightarrow{extension} \begin{pmatrix} d_0 & d_1 & \cdots & d_{N-1} \end{pmatrix} \begin{pmatrix} \hat{\phi}^0_{j,0}(x) \\ \hat{\phi}^0_{j,1}(x) \\ \vdots \\ \hat{\phi}^0_{j,N-1}(x) \end{pmatrix},$$

where $d_k = \sum_{l=0}^{N-1} C^0_{j,l} T^0_{l,k}$, $k = 0, \cdots, N-1$. And the point values of ϕ at scale j' can be obtained via the cascade algorithm [27].

Similarly, on the interval $[a+(b-a)(1-2^{-j}), b]$, $f \approx f_j = \sum_{k=0}^{N-1} C^1_{j,k} \phi^1_{j,k}(B^{-1}(x))$,

$$C^0_{j,k} = 2^{-j/2} \int f(B(1 - 2^{-j}x)) \phi^{0,\sharp}_{0,k}(x)dx.$$

It can be extended on the right edge as follows,

$$f_j = \begin{pmatrix} C^1_{j,0} & C^1_{j,1} & \cdots & C^1_{j,N-1} \end{pmatrix} T^1 \begin{pmatrix} \tilde{\phi}^1_{j,0}(x) \\ \tilde{\phi}^1_{j,1}(x) \\ \vdots \\ \tilde{\phi}^1_{j,N-1}(x) \end{pmatrix}$$

$$\xrightarrow{extension} \begin{pmatrix} e_0 & e_1 & \cdots & e_{N-1} \end{pmatrix} \begin{pmatrix} \hat{\phi}^1_{j,0}(x) \\ \hat{\phi}^1_{j,1}(x) \\ \vdots \\ \hat{\phi}^1_{j,N-1}(x) \end{pmatrix},$$

where $e_k = \sum_{l=0}^{N-1} C^1_{j,l} T^1_{l,k}$, $k = 0, \cdots, N-1$.

Remark 4.1.4. The lengths of extended intervals on each side are both $(b-a)(2N-2)/2^j$ which is smaller than $b-a$ since $j \geq J \geq 2N$.

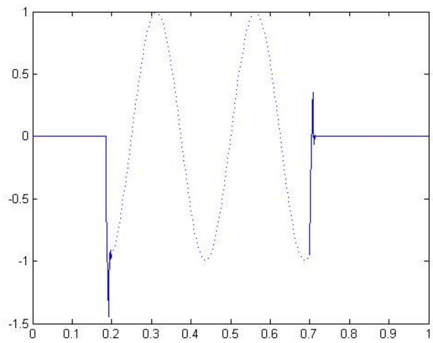
4.1.6 Exemples numériques

Univariate situation

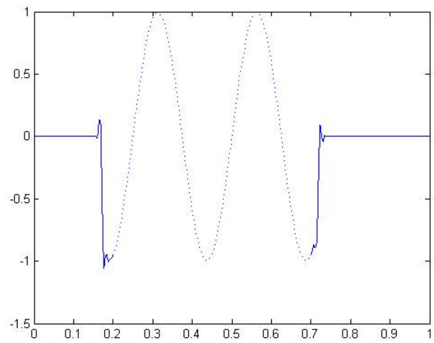
We consider the extension of the function $sin(8\pi x)$ from the interval $[0.2, 0.7]$ to $[0, 1]$ for different values N. The parameter used is $j = 6$. The plots are shown in figure 4.3 and figure 4.4 for different values of N. Figure 4.5 shows the extensions for $N = 10$ and different values of j.

Bivariate situation

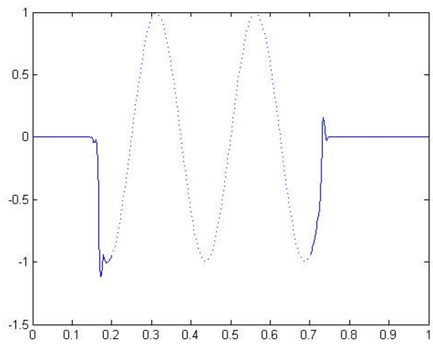
Here we consider the original domain $\omega = \{(x, y), \sqrt{(x-0.5)^2 + (y-0.5)^2} = 0.3\}$, $f(x, y) = sin(2xy)$, $j = 6$, $N = 5, 10$. The extension is showed on figure 4.6.



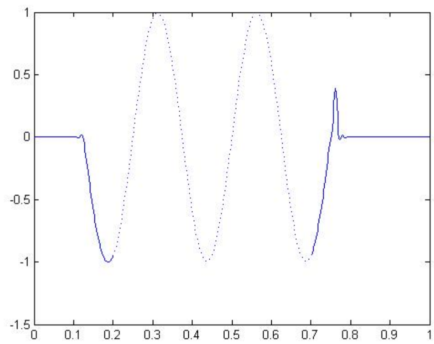
(a) N=2



(b) N=4

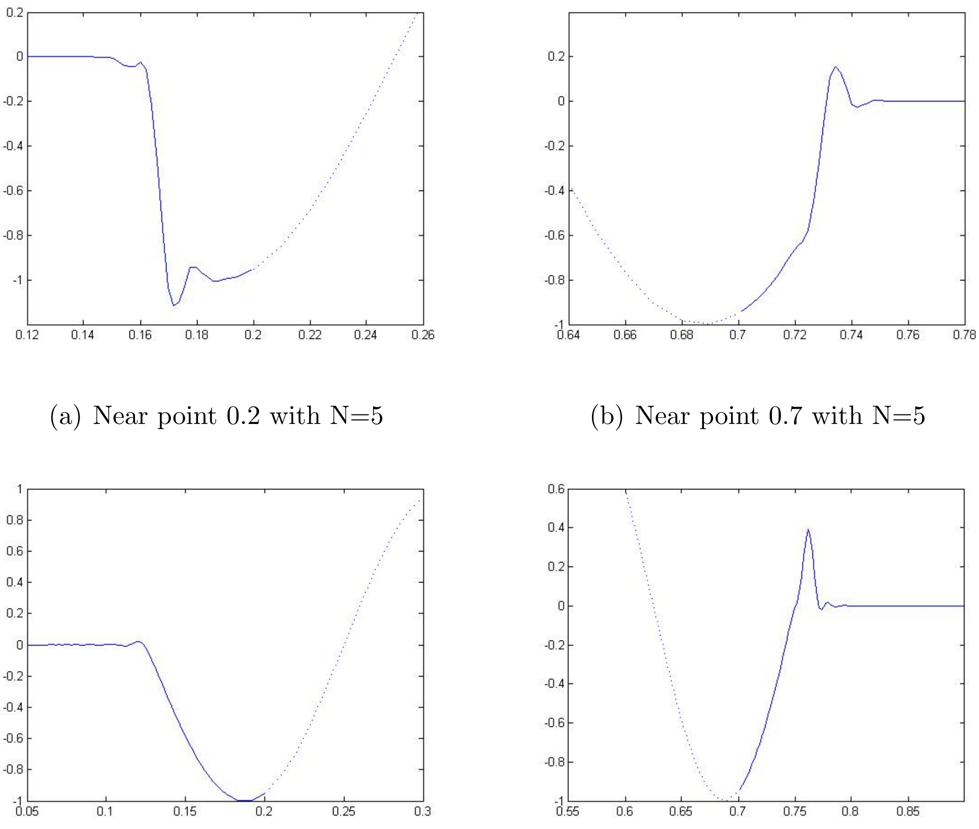


(c) N=5



(d) N=10

Figure 4.3: Extension of $sin(8\pi x)$ on the interval $[0.2, 0.7]$ with different N, the extension part is denoted by the solid line.



(a) Near point 0.2 with N=5 (b) Near point 0.7 with N=5

(c) Near point 0.2 with N=10 (d) Near point 0.7 with N=10

Figure 4.4: Zoom of the extensions near the points 0.2 and 0.7 with $N = 5$ or 10, the extension part is denoted by the solid line.

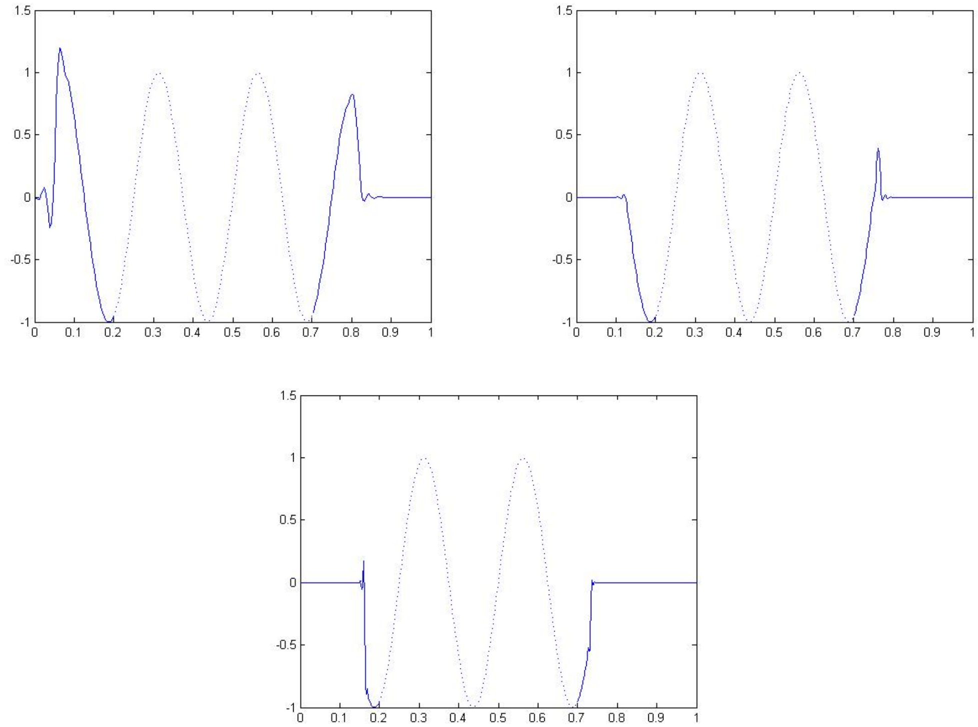


Figure 4.5: From top left, top right to bottom, $N = 10$,different extension with $j = 5, 6, 7$

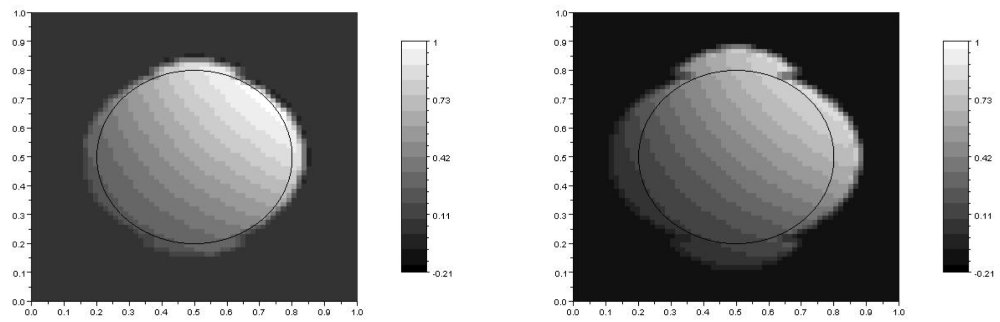


Figure 4.6: extension of $f(x, y) = sin(2xy)$ on a disk, left $N = 5$, right $N = 10$

4.2 Procédure de prolongement, pour une fonction connue sur une grille de points de ω

4.2.1 Introduction

A new construction of wavelets on $[0,1]$ by S.Bertoluzza et S. Falletta[10] overcomes the restriction of non interacting boundaries, i.e. $j \geq j_0 = log_2(2N)$, N the moments. If we use Daubechies scaling function ϕ with reproduction degree $M = N-1$, the largest scale is equal to 0. The restriction to $[0,1]$ of $\{V_j^*, j \in \mathbb{Z}\}$ is a multiresolution analysis of $V_j^{[0,1]}$. More precisely,

$$\begin{aligned} V_j^* = \{f = \sum_{k\in\mathbb{Z}} f_k^j \phi_{j,k} &: \exists p_l, p_r \in P_{M_j} \text{ such that} \\ f_k^j &= p_l(x), \forall k \in]-\infty, N-1] \\ f_k^j &= p_r(x), \forall k \in [2^j - N + 1, +\infty[\\ f_k^j &= (f, \phi_{j,k}), \text{ else} \end{aligned}$$

i.e. the coefficient has a polynomial behavior across the boundaries 0 and 1. Here $M_j = \min\{M, 2^j\}$, $N_j = 2^j + 1$ is respectively the dimension and polynomial exactness of V_j^*. The Riesz basis of $V_j^{[0,1]}$ is

$$\breve{\phi}_{j,k}^{[0,1]} = \sum_{m=-N}^{N+2^j} \eta_m^{j,k} \phi_{j,m}|_{[0,1]} \text{ with } \eta^{j,k} = (E_j \delta^{j,k})$$

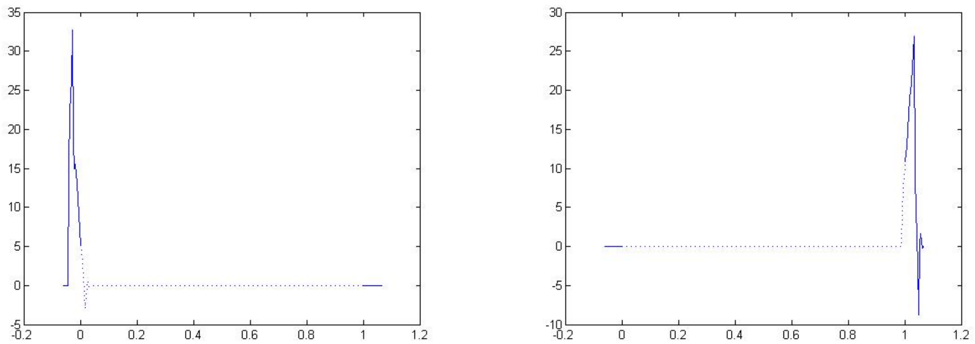
where E_j is an extension operator which extrapolate the $\{f_k^j\}_{k\in I_j}$ on the left and right by a polynomial of degree M_j(see [10]), $I_j = \{0, \cdots, 2^j\}$, $\delta^{j,k} = (\delta_n^{j,k})_{n\in I_j}$, and

$$\delta_n^{j,k} = \begin{cases} 0 \text{ if } n \neq k \\ 1 \text{ if } n = k \end{cases}$$

Given $p \in \mathbb{N}^*$, we construct the space V_p, similar to V_j^*, by replacing the 2^j by p.

Remark 4.2.1. The property of nestedness of $\{V_p\}_{p\in\mathbb{N}^*}$ does not hold any more, except for $p = 2^j, j \in \mathbb{Z}$.

We note that the $\check{\phi}_{j,k}^{[0,1]}$ has the same regularity as ϕ, i.e. $C^{0.2N}$. Removing the restriction to $[0,1]$, we has a natural extension of $\check{\phi}_{j,k}^{[0,1]}$. Take $N=2$, $j=6$ for example, see in figure 4.7. Denote the extension part by solid line.



(a) $\check{\phi}_{6,0}^{[0,1]}$ (b) $\check{\phi}_{6,64}^{[0,1]}$

Figure 4.7: extension of Riesz basis

4.2.2 De l'intervalle de [0,1] à $I\!R$

For function $f \in V_p$,

$$f|_{[0,1]} = \sum_{k \in I_j} (\int_{I\!R} f \phi_{j,k}) \check{\phi}_{j,k}^{[0,1]}.$$

If $\check{\phi}_{j,k}^{[0,1]}$ is extended by removing the restriction to $[0,1]$, the function f is smoothly extended accordingly f_{ex} with compact support $[-\frac{2N}{p}, \frac{2N+p}{p}]$. The coefficient $\int_{I\!R} f\phi_{j,k}$ is calculated as follows:

(1) $p \leq N-1$. Note that under this situation, the space $V_j^{[0,1]}$ coincides with the space of polynomial of degree $M_j = p$. Let f_p be the polynomial assuming the values $c_0, c_1 \cdots, c_p$ at the points $0, 1/p, \cdots, 1$, $c_i = f(\frac{i}{p})$, $i = 0, \cdots, p$ i.e.

$$f_p = \sum_{m=0}^{p} c_m L_{p,m}(x),$$

with

$$L_{p,m}(x) = \prod_{\substack{i=0 \\ i \neq m}}^{p} \frac{px - i}{m - i}.$$

Then the coefficient

$$\int_{I\!R} f\phi_{j,k} = \int f_p \phi_{j,k}.$$

It will be obtained by $\int_{I\!R} x^n \phi$, $n = 0, \cdots, p$.

(2) $p > N-1$. From the $p+1$ point values on the interval $[0,1]$, we extrapolate these point values at the edge by polynomial of order M_j. So the constructed function has a polynomial behavior across the edge, therefore in the space V_p. Then we use the quadrature formula (3.22) of four order to calculate the scaling coefficient.

Remark 4.2.2. If $p = 0$, i.e. there is only one point value known on $[0,1]$. We use $\sum_n \phi(x-n) = 1$ to do smooth extension. The extended function f_{ex} has a compact support $[-2N, 2N]$.

4.2.3 D'un intervalle arbitraire [a,b] à $I\!R$

Let $\{a, a+\frac{b-a}{p}, \cdots, b\}$ be the $p+1$ equal distant points on $[a,b]$ assuming values $d_0, d_1, \cdots, d_p$, $d_i = f(a + i\frac{b-a}{p})$, $i = 0, \cdots, p$. Then one uses B^{-1} to map $\{a, a+\frac{b-a}{p}, \cdots, b\}$ to $\{0, \frac{1}{p}, \cdots, 1\}$. From the points $\{(0,d_0), (\frac{1}{p}, d_1), \cdots, (1, d_p)\}$ we do smooth extension on $[0,1]$. The extension function is f^*_{ex}. So the smooth extension f_{ex} of original problem is $f_{ex}(Bx) = f^*_{ex}(x)$ with compact support $[a-(b-a)\frac{2N}{p}, b+(b-a)\frac{2N}{p}]$.

4.2.4 D'un domaine convexe $\omega \subset I\!R^2$ à $I\!R^2$

Here we consider the problem of the smooth extension of function f from any convex domain $\omega \subset I\!R^2$ to $I\!R^2$ through its pointvalues on a regular grid of scale j, i.e. $\{f(k2^{-j}, k'2^{-j}), (k,k') \in \Lambda_\omega\}$, where Λ_ω denotes all the index (k,k') such that $(k2^{-j}, k'2^{-j}) \in \omega$.

For any N and given a compact support scaling function having N moments, the extension is performed as follows(see figure 4.8):

(a) On each horizontal line $y = k'2^{-j}$, determine the segment $[a,b]$ of maximum length, $a = k_1 2^{-j}$,$b = k_2 2^{-j}$, where $(k_1,k'), (k_2,k') \in \Lambda_\omega$, then construct the approximation space $V_p([a,b])$ from the pointvalues $\{f(k2^{-j}, k'2^{-j}), k = k_1, \cdots, k_2\}$.

(b) Proceed the smooth extension from $V_p([a,b])$ to $V_p(R)$ on each line.

(c) On each column proceed the smooth extension.

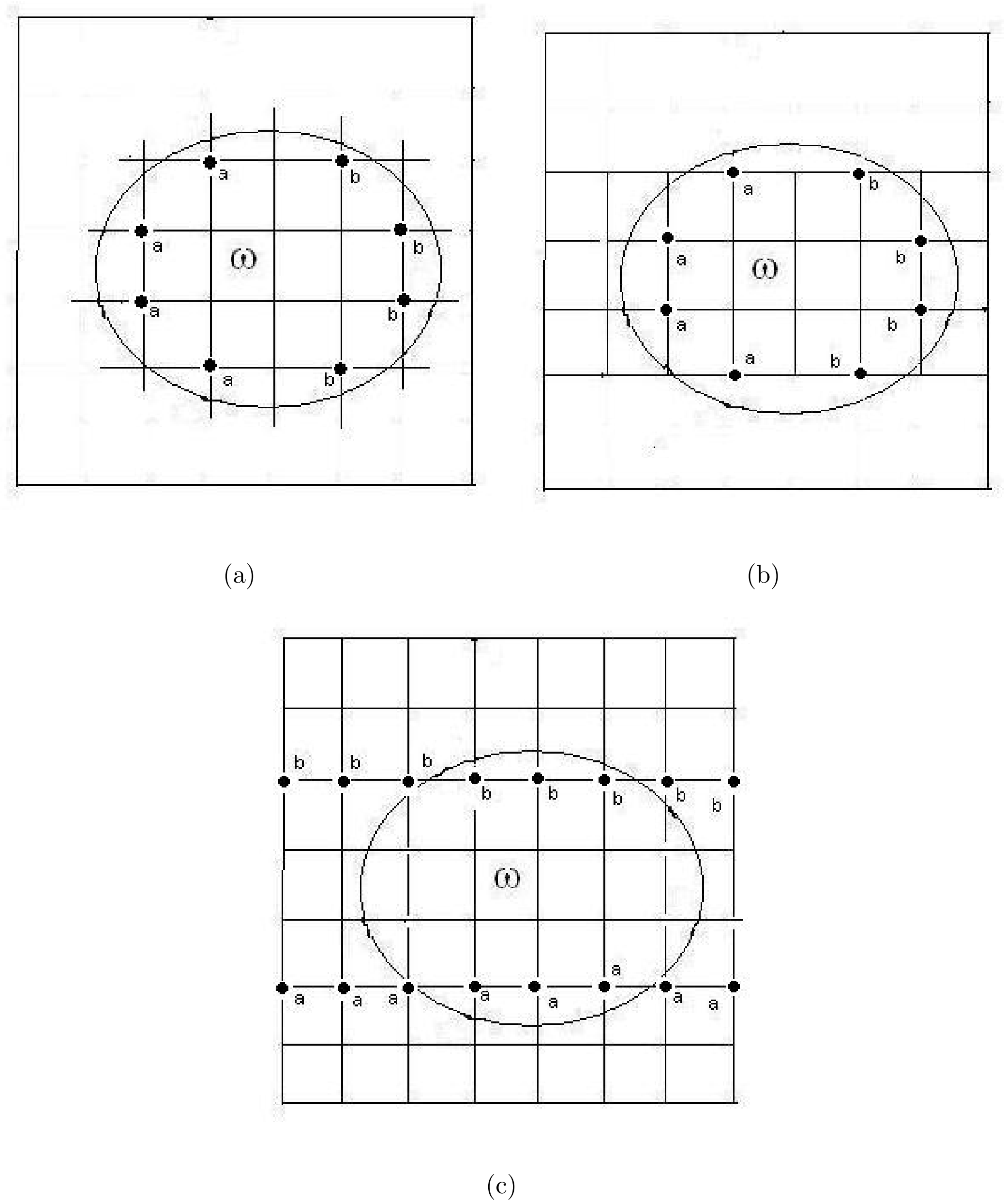
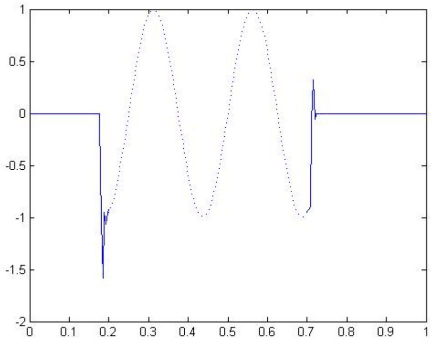


Figure 4.8: extension procedure

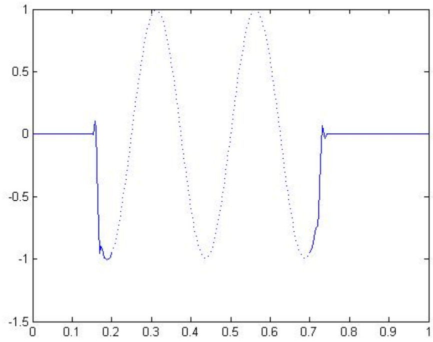
4.2.5 Exemples numériques

Univariate situation

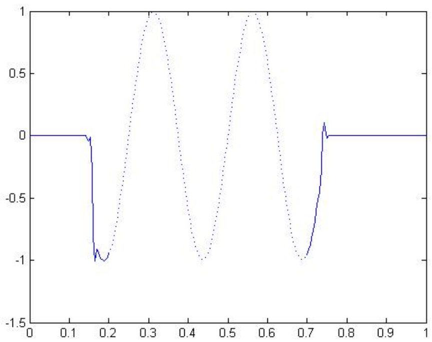
We consider to extend the function $sin(8\pi x)$ from the interval $[0.2, 0.7]$ to $[0, 1]$ for different values N. The plots are shown on figure 4.9 for $p = 2^6$. For fixed $N = 10$, we choose different p to see its extension on figure 4.10.



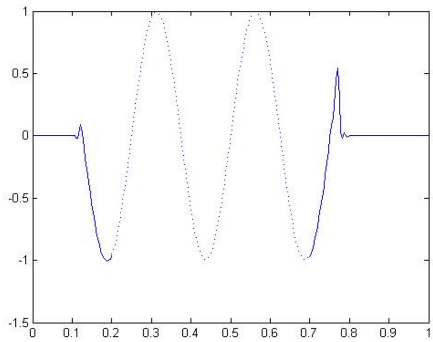
(a) N=2



(b) N=4



(c) N=5

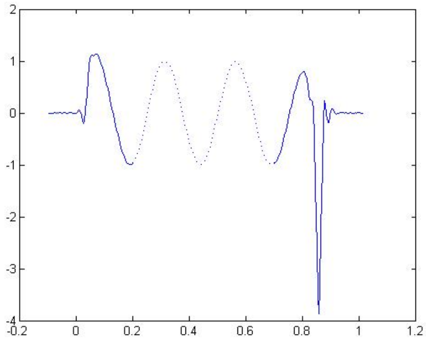


(d) N=10

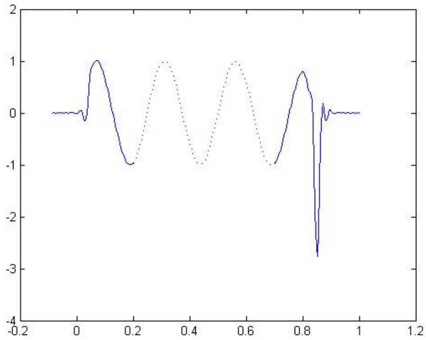
Figure 4.9: Extension of $sin(8\pi x)$ on the interval $[0.2, 0.7]$ with different N, the extension part is denoted by the solid line.

Bivariate situation

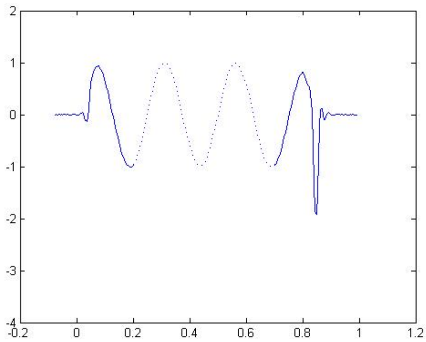
Here we consider the original domain $\omega = \{(x, y), \sqrt{(x - 0.5)^2 + (y - 0.5)^2} = 0.3\}$, the point values(of function $f(x, y) = sin(2xy)$) in this disk on the grid of scale $j = 6$ are known. The point values on the grid in $\Omega \setminus \omega(\Omega = [0, 1] \times [0, 1])$ are



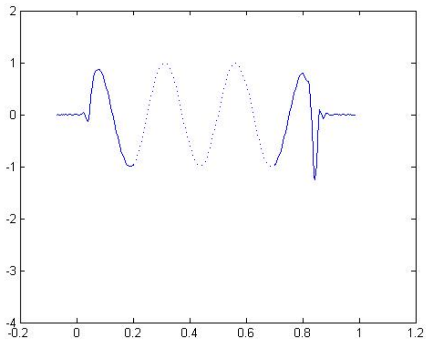
(a) $p = 2^5$



(b) $p = 2^5 + 1$



(c) $p = 2^5 + 2$



(d) $p = 2^5 + 3$

Figure 4.10: Extension of $sin(8\pi x)$ on the interval $[0.2, 0.7]$ with different p when $N = 10$, the approximation error in norm $L^2[a, b]$ is 2.886134e-002,2.772000e-002,2.637102e-002,2.562554e-002,respectively.

0. The figure 4.11 shows the extension in the x direction. And the extension in y direction is illustrated on figure 4.12.

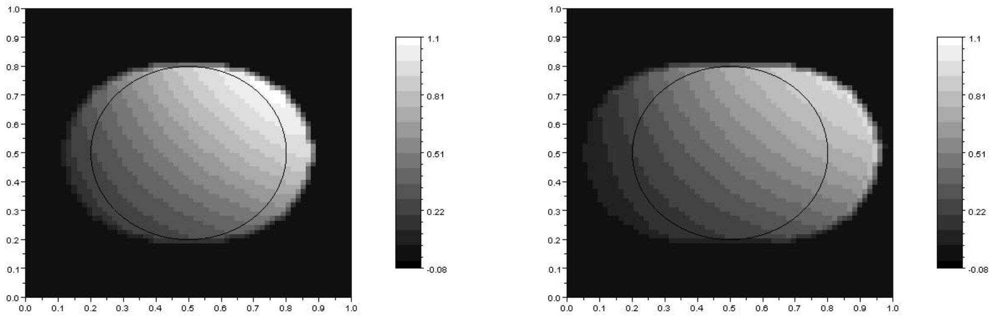


Figure 4.11: extension in x direction, left: $N = 5$, right: $N = 10$

Remark 4.2.3. After the extension in x direction, the function we obtained is of poor continuity in the y direction which will affect the approximation ability of wavelet.

For the remark above, one solution is to replace the procedure (b) by taking the smallest a and largest b as common(see on figure 4.13). For the point outside the domain ω, its value is known through the Lagrange interpolation with polynomial of order M_j at each side on the horizontal line. So the constructed function with polynomial behavior across the edge is still in V_j^*. With this method, the extension is on figure 4.14.

4.3 Application à la méthode de domaine fictif lisse

4.3.1 Exemple en dimension 1

We consider the problem

$$\begin{cases} (I - \nu\triangle)u = f, \text{ on }]a, b[\\ \text{Dirichlet boundary condition at } a \text{ and } b. \end{cases}$$

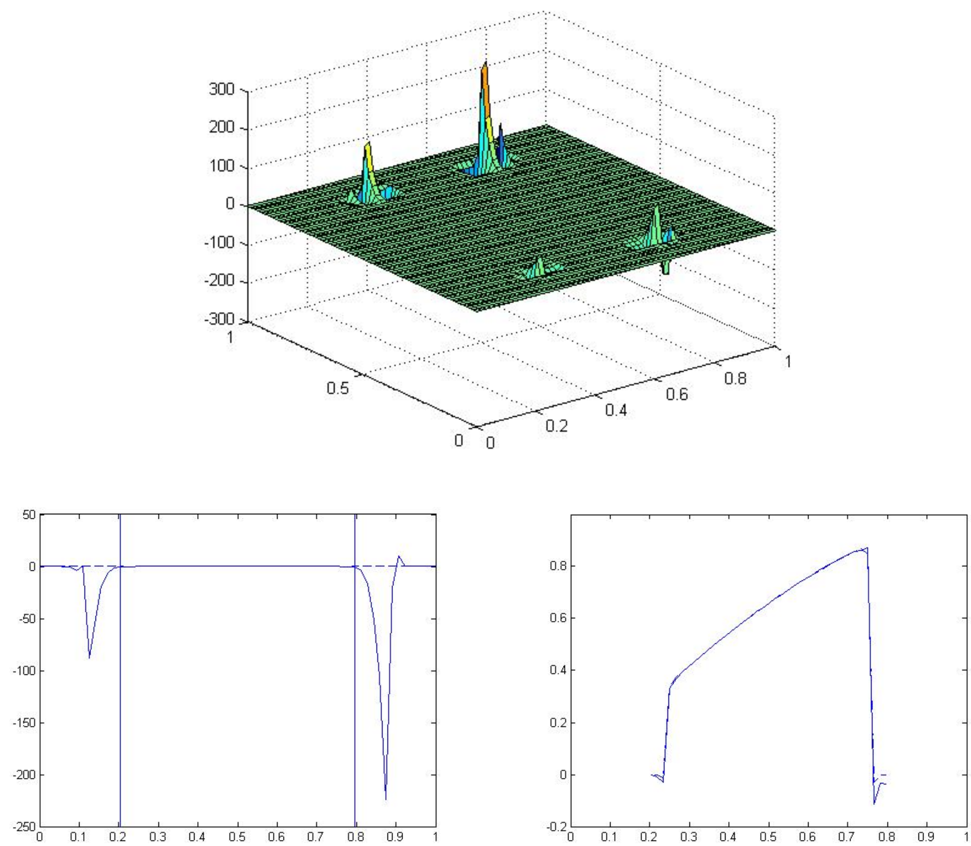


Figure 4.12: $N = 5$, top: plot after the extension in y direction, bottom: extension in y direction at $x = 0.71875$, left: $y \in [0, 1]$, the two lines denotes the original domain, right: the approximation in the original domain, dashed line for original function, solid line for approximation function.

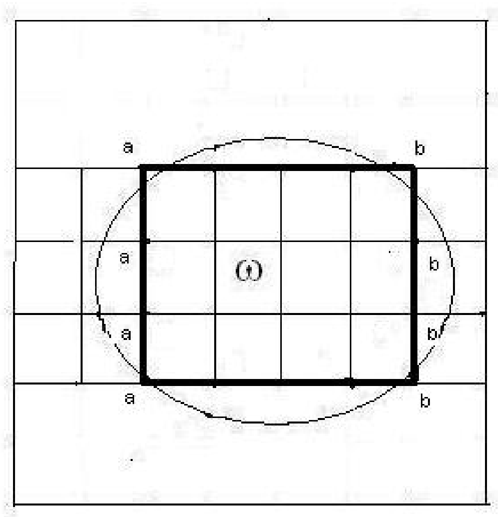


Figure 4.13: modified procedure b

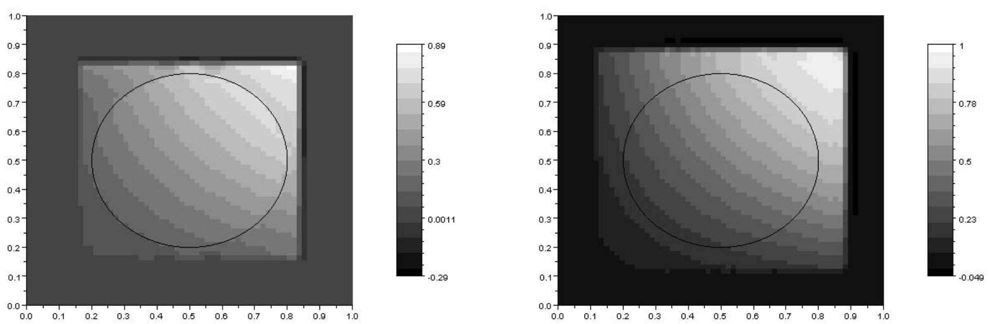


Figure 4.14: left: $N = 2$, right: $N = 5$

Here the exact solution $u(x) = x^5 - x^3 + 12x^2 - 2.5x + 4$, $f = (I - \nu\triangle)u$ in $[a, b]$, $\nu = \frac{1}{64\pi^2}$, $a = 0.2$, $b = 0.7$. We choose a spline order $m = 6$. Then we compare the error in norm $L^2[a, b]$ or $H^1[a, b]$ in tree different cases, i.e. old fictitious method with $a' = a$, $b' = b$ and f is constantly extended, smooth fictitious method with $a' = 0.1$, $b' = 0.9$, f is C^1 extended with $N = 5$, smooth fictitious method with $a' = 0.1$, $b' = 0.9$, f is C^2 extended with $N = 10$. The results are illustrated in figure 4.15, 4.16.

From the analysis for the old fictitious domain method, we know that the maximum convergence rate in norm $L^2[a, b]$ is $-\frac{3}{2}log(2)$ while it is $-\frac{1}{2}log(2)$ in norm $H^1[a, b]$ using the semi-logarithmic scale since the derivative jump is just on points a and b. For the smooth method, we see that a better rate of convergence can be reached in the interior domain as soon as the extended function is sufficiently smooth on this domain. If f is C^1(resp C^2) extended, it follows $\bar{u} \in H^3([a', b'])$(resp $\bar{u} \in H^4([a', b'])$). According to theorem 3.22 (interior estimates for univariate situation), the maximum convergence rate is of $-3log2$(resp $-4log2$) for error in norm $L^2[a, b]$ and of $-2log2$(resp $-3log2$) for error in norm $H^1[a, b]$ using the semi-logarithmic scale.

Remark 4.3.1. For the old fictitious domain method, it does not increase the convergence rate when f is smoothly extended(see on figure 4.17).

4.3.2 Exemple en dimension 2

Here $\omega = \{(x, y); \sqrt{(x - 0.5)^2 + (y - 0.5)^2} = 0.2\}$, $\Xi = \{(x, y); \sqrt{(x - 0.5)^2 + (y - 0.5)^2} = 0.4\}$, $u(x) = sin(\pi x) + sin(\pi y) + cos(\pi x) + cos(\pi y) + x^6 + y^6$, $f = (2I - 3\triangle)u$ in ω, spline order $m = 4$ and C^1 extended with $N = 5$ or C^2 extended with $N = 10$ in $\Xi \setminus \omega$, $m = 6$, $j' = j'' = 4$. The plots 4.18, 4.19 show the decrease rate of error in norm $L^2(\omega)$ or $H^1(\omega)$ versus scale j in semi-logarithmic scale.

If f is C^1(resp C^2) extended, it follows $u \in H^3(\Xi)$(resp $u \in H^3(\Xi)$). According to theorem 3.23, the maximum convergence rate of error in norm $H^1(\omega)$ is $-2log2$(resp

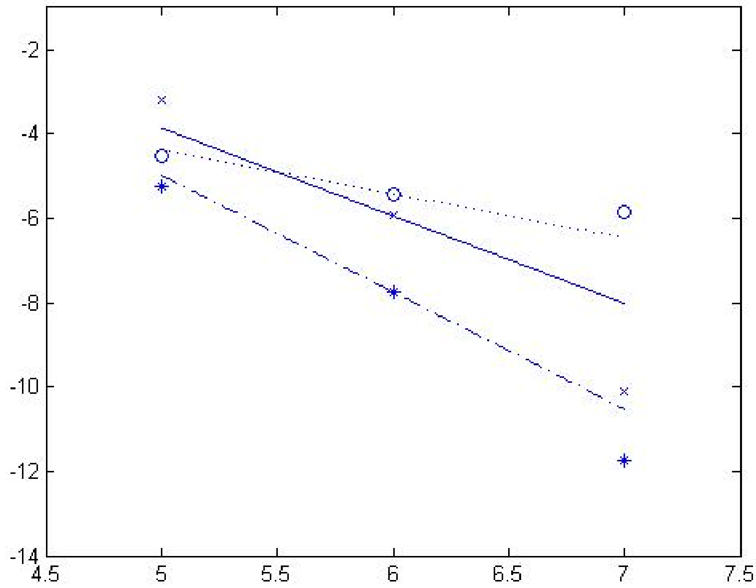


Figure 4.15: Error in norm $L^2[a,b]$ versus j in semi-logarithmic scale, O: old method, X: smooth method with $N = 5$, *: smooth method with $N = 10$, dotted line of slope $-\frac{3}{2}log2$, solid line of slope $-3log2$, dashed dot line of slope $-4log2$.

$-3log2$). And we expect the maximum convergence rate of error in norm $L^2(\omega)$ is $-3log2$(resp $-4log2$).

4.3.3 Comparaison entre l'extension de Fourier et ondelettes en 1D

Firstly we give a brief description of the implementation of Fourier extension developed in [30] in 1D. Let $[0, L]$ be an interval which is equally divided with level $h = \frac{L}{N}$, where N is an integer. Denote the known point value on hj by f_j, $j \in J$ while the unknown point value on hj' by $g_{j'}$, $j' \in J'$, $J \cup J' = \{0, 1, \cdots, N-1\}$. The extended function is $\tilde{f}$ which has the same value on the points hj, $j \in J$. So its discrete Fourier form reads:

$$\hat{\tilde{f}}_k = \frac{1}{N}(\sum_j f_j e^{-\frac{i2\pi kj}{N}} + \sum_{j'} g_{j'} e^{-\frac{i2\pi kj'}{N}}) \tag{4.2}$$

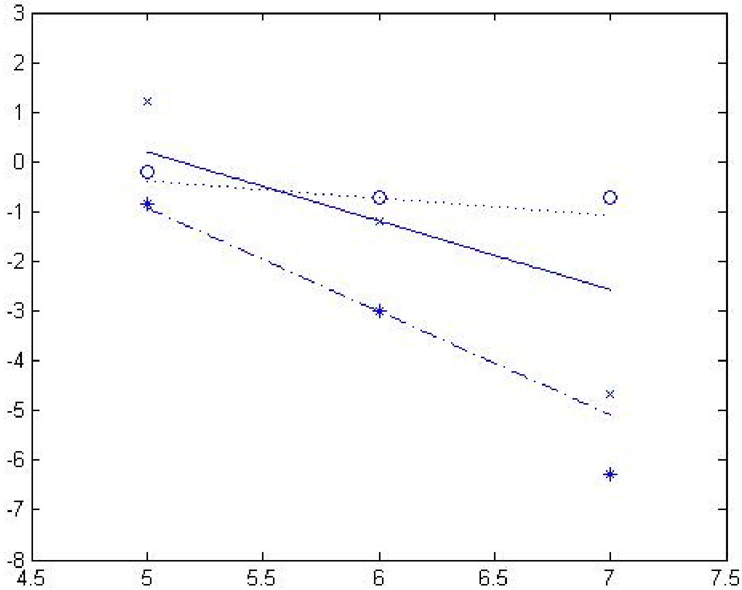


Figure 4.16: Error in norm $H^1[a,b]$ versus j in semi-logarithmic scale, O: old method, X: smooth method with $N=5$, *: smooth method with $N=10$, dotted line of slope $-\frac{1}{2}log2$, solid line of slope $-2log2$, dashed dot line of slope $-3log2$.

And the function $\tilde{f}$ is the one that minimize $\| \triangle^p \tilde{f}\|_{L^2}$ over all the possible values of $g_{j'}$. The index p is used to measure the regularity of $\tilde{f}$. From parseval identity, we know that

$$\begin{aligned} \| \triangle^p \tilde{f}\|_{L^2} &= \sum_k k^{2p}|\hat{\tilde{f}}_k|^2 \\ &= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \frac{k^{2p}}{N^2} | \sum_j f_j e^{-\frac{i2\pi kj}{N}} + \sum_{j'} g_{j'} e^{-\frac{i2\pi kj'}{N}}|^2 \end{aligned} \tag{4.3}$$

So the $g_{j'}$, $j' \in J'$ is determined in such a way that $\frac{\partial \|\triangle^p \tilde{f}\|_{L^2}}{\partial g_{l'}} = 0, \forall l' \in J'$. Namely, it satisfies

$$\sum_{j' \in J'} g_{j'} \sum_k k^{2p} e^{-i2\pi k(j'+l')} = -\sum_k k^{2p} (\sum_j f_j e^{-\frac{-i2\pi kj}{N}}) e^{-\frac{-i2\pi kl'}{N}}, \forall l' \in J'$$

The extension problem turns to resolve a linear system $Ax=b$, where

$$A_{l'j'} = \sum_k k^{2p} e^{-i2\pi k(j'+l')} \tag{4.4}$$

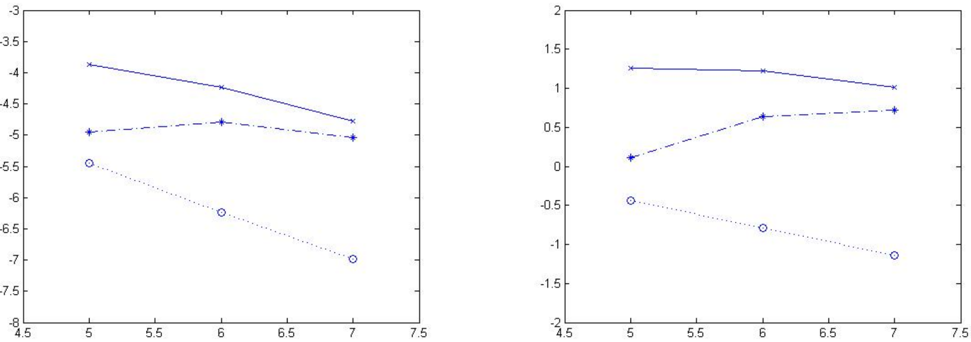


Figure 4.17: Left: error in norm $L^2[a,b]$ versus j in semi-logarithmic scale, right: error in norm $H^1[a,b]$ versus j in semi-logarithmic scale. O: old method with f constantly extended, X: old method with $N = 5$, *: old method with $N = 10$.

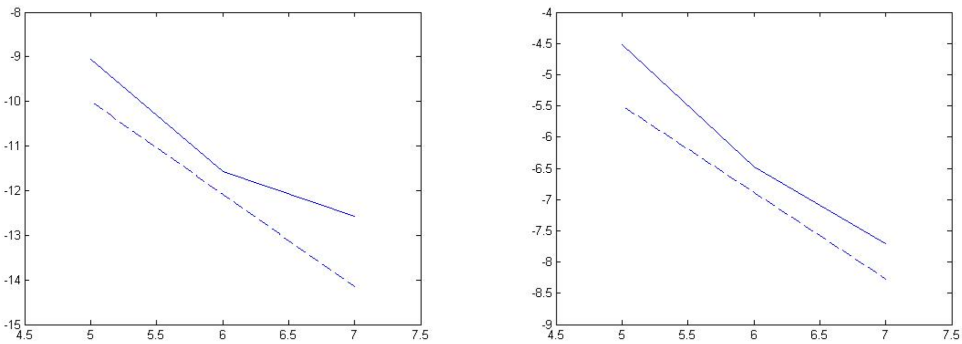


Figure 4.18: Left: solid line stands for error in norm $L^2(\omega)$ versus j in semi-logarithmic scale, dashed line with slope $-3log2$; right: solid line stands for error in norm $H^1(\omega)$ versus j in semi-logarithmic scale, dashed line with slope $-2log2$

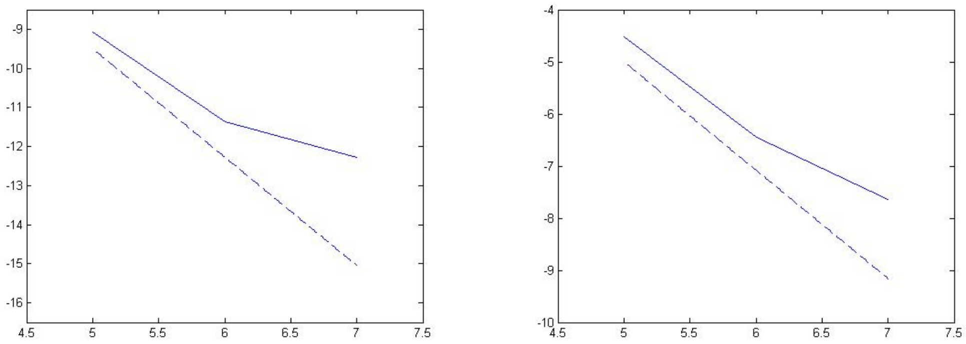


Figure 4.19: Left: solid line stands for error in norm $L^2(\omega)$ versus j in semi-logarithmic scale, dashed line with slope $-4log2$; right: solid line stands for error in norm $H^1(\omega)$ versus j in semi-logarithmic scale, dashed line with slope $-3log2$

$$b_{l'} = -\sum_k k^{2p} (\sum_j f_j e^{-\frac{-i2\pi kj}{N}}) e^{-\frac{-i2\pi kl'}{N}} \tag{4.5}$$

We use same example in section 4.3.1 to compare this two smooth extension techniques applied in the smooth fictitious domain method. The plots in semi-logarithmic scale are plotted on figure 4.20 and 4.21. We see that the wavelet extension method obtains a better convergence rate from $j = 6$ to 7 on the former plot and a smaller error on the latter plot compared to Fourier method.

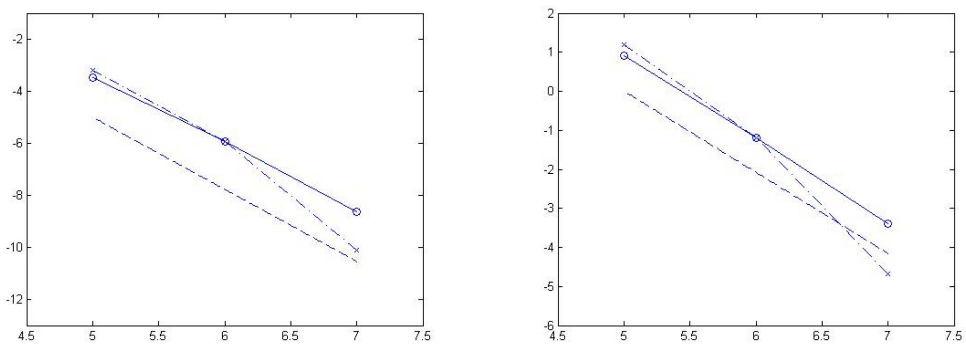


Figure 4.20: Left plot: norm in $L^2(\omega)$, dashed line of slope of $-4log2$. Right plot: norm in $H^1(\omega)$, dashed line of slope of $-3log2$. Solid line for Fourier method with $p = 1$; dashed dot line for wavelet method with $N = 5$.

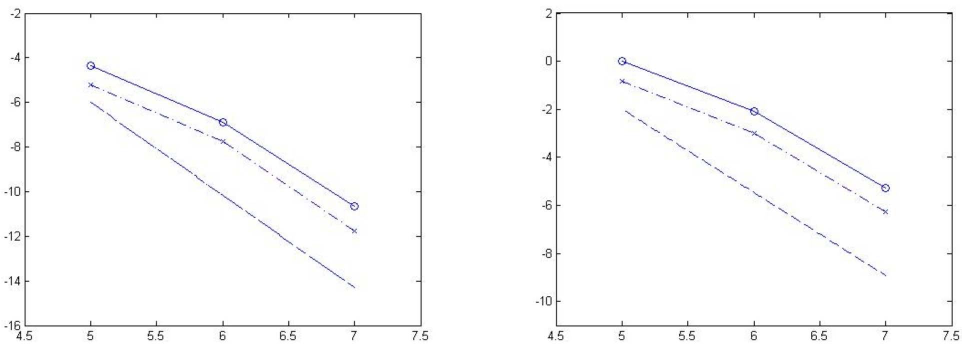


Figure 4.21: Left plot: norm in $L^2(\omega)$, dashed line of slope of $-6log2$. Right plot: norm in $H^1(\omega)$, dashed line of slope of $-5log2$. Solid line for Fourier method with $p = 2$; dashed dot line for wavelet method with $N = 10$.

Chapter 5

Simulation numérique du problème de Stefan à deux phases

Le problème de Stefan que nous considérons ici est composé de 2 équations paraboliques (une sur chaque domaine ω et $\Omega\backslash\omega$) et d'une équation hyperbolique pour la frontière de ω. Nous explicitons d'abord la discrétisation du problème: En utilisant la méthode d'Euler implicite, le problème se transforme en une séquence de problèmes elliptiques qui sont approchés par la méthode de domaines fictifs lisses. La frontière mobile est capturée par la méthode Level Set et son équation est approchée par un schéma de type différences finies. Les pas de temps utilisés pour la discrétisation des équations paraboliques et de l'équation de Level Set sont différents et résultent de l'analyse des erreurs et de la condition CFL respectivement. Différents exemples sont considérés afin de valider l'efficacité de notre méthode. Ensuite, nous appliquons cette méthode à un problème de Stefan à deux phases avec condition aux limites de type Gibbs-Thomson.

The two phase Stefan problem modelizes the evolution of the temperature field of the mixing of two phases and the evolution of the boundaries between the two phases. This problem is typical of the melting of ice in water, crystal growth or dendritic solidification. From a physical point of view, the influence of the parameters on the shape of the front is of great interest for scientists in disciplines such as chemistry, geology, materials science. M. Hinze and S. Ziegenbalg [42], for instance, try to control the evolution of the boundary using the temperature on the container wall.

There are many different numerical approaches to capture the moving boundary. Boundary integral methods[11] are used for 2D diffusion dominated problems, but are difficult to apply for higher dimension. Front tracking methods[44] such as presented by D. Juric and G. Tryggvason explicitly track the liquid-solid interface. They successfully cope with the two-dimensional Stefan problem with complex dendritic structure. However, the drawback of the methods is that they hardly manage topological changes of the fronts. The level set method is a good choice to overcome the two difficulties listed above. It has been introduced by S. Osher and J.A. Sethian in [58]. The main idea behind the level set method is that the front is always represented by the zero level set of a smooth, continuous function. Hence, the front can be plotted simply by a specific contour level. For example, in [14] the front is described as a level set of a signed distance function.

Generally, the evolution of temperature is governed by the heat equations on each phase with a coupling through the boundary conditions. As far as the theoretical analysis is concerned, we refer the reader to [49],[62]. In most cases, we use the Dirichlet, Neumann boundary conditions. However other boundary conditions can also be applied according the physical problems considered, for example, in [59] the authors use the Robin boundary conditions. Concerning approximation, various methods have been developped, including finite difference method [35][34], finite element method [57][64] and wavelet method [4].

5.1 Introduction

In this work, we consider the two-phase Stefan problem in a square domain Ω. The two phases, typically solid and liquid denoted by ω and $\omega^c = \Omega \backslash \omega$, are separated by an interface γ. Let $T(\vec{x}, t)$ and V separately represent the temperature of the phases and the normal velocity at the front γ. The governing equations are formulated as follows,

$$\begin{cases} \dfrac{\partial T}{\partial t} = \kappa \triangle T, \quad \vec{x} \in \Omega, \\ V = \kappa [\bigtriangledown T]|_\gamma \cdot \vec{n}. \end{cases} \tag{5.1}$$

Here κ is the diffusion coefficients which may be different on each phase, the jump is taken from solid to liquid, the vector $\vec{n}$ is the outward normal vector at the front and $[\nu]|_\gamma$ stands for the discontinuity of ν along the normal of boundary γ. In melting, one usually sets as boundary conditions, $T(\vec{x}, t) = T_m$ on γ where T_m is a constant equal to the melting temperature of the material. In the case of crystal growth and dendritic solidification, one considers the Gibbs-Thomson relation:

$$T(\vec{x}, t) = -\epsilon_C k - \epsilon_V V,$$

where k denotes the curvature at the front, ϵ_C the surface tension coefficient, and ϵ_V the molecular kinetic coefficient.

Here we use level set function η defined on Ω, which denotes the signed distance from the boundary γ, to track the moving boundary. It is initialized by η_0 which reads

$$\eta_0 = \begin{cases} d(x, \gamma), \quad \vec{x} \in \omega^c, \\ 0, \quad \vec{x} \in \gamma, \\ -d(x, \gamma), \quad \vec{x} \in \omega, \end{cases}$$

where d is the distance from the boundary, i.e., $d(x, \gamma) = \inf\{||x - y||_2, y \in \gamma\}$. Then, the third equation in (5.1) is replaced by an equation of η.

$$\begin{cases} \eta_t + V \bigtriangledown \eta = 0 \\ \eta \,|_{t=t_0} = \eta_0 \end{cases} \tag{5.2}$$

where V is an extension of the velocity on the boundary on Ω(of same notation) and note $V = (V_x, V_y)$. So the system (5.1) is rewritten,

$$\begin{cases} \dfrac{\partial T}{\partial t} = \kappa \triangle T, \quad \vec{x} \in \Omega, \\ T \mid_{t=t_0} = T_0, \\ \text{boundary condition of } T, \quad \vec{x} \in \gamma \\ \eta_t + V \bigtriangledown \eta = 0, \quad \vec{x} \in \Omega \\ \eta \mid_{t=t_0} = \eta_0. \end{cases} \tag{5.3}$$

It is composed of two coupled parabolic problem, one raised on ω, the other on ω^c which will be discretized in time by the backward Euler method, which is first order accurate with respect to time step δt_p, to get a series of elliptic equations of form:

$$(1 - \delta t \kappa \triangle) T^{n+1} = T^n, \tag{5.4}$$

where T^n stands for the approximation of $T(t_n, \cdot)$, $t_n = n\delta t_p$, $0 \leq t_n \leq T$, $n \in Z$. This elliptic problem has been treated in previous chapiter with the smooth fictitious domain method, Petrov-Galerkin method and smooth extension technique. Let T_j^n be the approximate solution of T^n at level j and λ^n be the Lagrange multiplier, the error of the parabolic equation reads: If $T^n \in H^s(\omega)$, $s \leq n$, $\lambda^n \in H^{s'}(\omega)$, $s' \leq n'$[6]

$$\begin{aligned} \|T(t_{n+1}, \cdot) - R_\omega T_j^{n+1}\|_{L^2(\omega)} \leq \|T(t_0, \cdot) - T^0\|_{L^2(\omega)} + \|T_j^0 - T^0\|_{L^2(\Omega)} \\ + cT\delta t_p + \frac{c'T}{\delta t_p}(2^{-js} + 2^{-j's'}) \end{aligned}$$

where c and c' are two constants, j' is the discretization level on γ, n (resp n') is the order of multiresolution on Ω(resp γ). The problem governing the moving boundary is an hyperbolic equation. There exist various methods(see in [43])to solve it, including TVD method, ENO method. In sum, we need to alternatively solve these three problems.

Finally, on can summarize the different problems related to the two phase Stefan problem: (1) the discretization of the equation for the boundary; (2) evaluation of the velocity on the boundary and its extension to the domain around; (3) calculation of the moving boundary; (4) choice of the time step for the equation of moving boundary(CFL condition, named after Courant, Friedrichs and Lewy).

5.2 Discrétisation du problème

We take the domain Ω to be $[0,1]\times[0,1]$ in our computation and discretize it at scale l with mesh size $h=\frac{1}{2^l}$ and the diffusion coefficient $\kappa=1$ and note

$$\begin{aligned}
&\vec{x}_{i,j}=((i-1)h,(j-1)h),\\
&\eta_{i,j}=\eta(\vec{x}_{i,j}),\\
&T_{i,j}=T(\vec{x}_{i,j}),\\
&i,j=1,\cdots,2^l+1.
\end{aligned}$$

5.2.1 Discretisation en temps

We choose two different time steps of discretization, one for the parabolic equation which determine the temperature evolution, the other for the hyperbolic equation which captures the moving boundary. For the former, it is restricted to $\delta t_p = ch$, $0<c<1$. For the latter, the time step δt_h should satisfy the CFL condition which reads

$$\delta t_h < \min\{\frac{h}{\max|V_x|},\frac{h}{\max|V_y|}\}$$

Usually, δt_p is greater than δt_h.

5.2.2 Calcul du gradient de température

The temperature on the grid is approximated by a function in $V_l(\Omega)$ spanned by periodic spline scaling functions, i.e. $P_{V_l(\Omega)}T(x,y)=\sum_{0\le k_1,k_2<2^l} c^l_{k_1,k_2}\Phi^{\Omega}_{l,k_1,k_2}$, where $P_{V_l(\Omega)}$ is an orthogonal projector. Its coefficient $c^l_{k_1,k_2}$ with scaling function is obtained through the interpolation process:

$$\begin{array}{c} T(s2^{-l}, t2^{-l})_{0\le s,t<2^l} \\ \big\downarrow{\scriptstyle FFT} \\ \hat{T}(k_1,k_2)_{0\le k_1,k_2<2^l} \\ \big\downarrow \\ \hat{c}^l(k_1,k_2) = 4^l \hat{T}(k_1,k_2)\hat{L}_l(k_1)\hat{L}_l(k_2) \\ \big\downarrow{\scriptstyle FFT^{-1}} \\ c^l_{k_1,k_2} \end{array}$$

The discrete fourier transform and its inverse form in 2D are,

$$\hat{T}(k_1,k_2) = \frac{1}{N^2}\sum_{s,t=0}^{N-1} T_{s,t} e^{-2i\pi(sk_1+tk_2)/N},$$

$$T_{s,t} = \sum_{k_1,k_2=0}^{N-1} \hat{T}(k_1,k_2) e^{2i\pi(sk_1+tk_2)/N}.$$

where $N = 2^l$, $\hat{L}_l$ see in section 1.8.3. The gradient of temperature is calculated through the derivative of B-spline.

5.2.3 Calcul de la courbure

For the boundary condition, it is noted that the interface curvature is performed with standard second order accurate central differencing on grid point neighboring the interface and then linearly interpolated. It hinds the tree order accuracy of their method for the Stefan problem [35][34]. We will propose a remedy using the property of B-spline. In two dimension, One knows that the signed curvature k of a curve defined as $c(t) = (x(t), y(t))$ is

$$k = \frac{x'y'' - y'x''}{((x')^2 + (y')^2)^{3/2}}.$$

We use the level set function η to capture the moving interface γ as in [14]. The front is equal to the zero level set of η, i.e.

$$\gamma(t) = \{\vec{x} \in \Omega, \eta(\vec{x}, t) = 0\},$$

The first and second order derivative of $\eta(x,y)=0$ in t are separately,

$$\eta_x x_t + \eta_y y_t = 0,$$

$$(\eta_{xx}x_t + \eta_{xy}y_t)x_t + (\eta_{xy}x_t + \eta_{yy}y_t)y_t + \eta_x x_{tt} + \eta_y y_{tt} = 0.$$

From the two equalities above, one gets the curvature

$$k = \frac{\eta_y^2\eta_{xx} - 2\eta_x\eta_y\eta_{xy} + \eta_x^2\eta_{yy}}{(\eta_x^2+\eta_y^2)^{3/2}}.$$

We use the interpolation process introduced above to approximate η by a function in space $V_l(\Omega)$, i.e.

$$P_{V_l(\Omega)}\eta = \sum_{0\le k_1,k_2<2^l} c^l_{k_1,k_2}\Phi^{\Omega}_{l,k_1,k_2},$$

The curvature is obtained by the calculation of derivative of B-spline in one or two order.

5.2.4 Vitesse et prolongement

The velocity in x and y direction on the domain Ω are

$$V_x|_{\vec{x}_{i,j}} = -S(\eta_x)|_{\vec{x}_{i,j}}(\frac{\partial T}{\partial x}|_{\vec{x}_{i+1,j}} - \frac{\partial T}{\partial x}|_{\vec{x}_{i-1,j}}),$$

$$V_y|_{\vec{x}_{i,j}} = -S(\eta_y)|_{\vec{x}_{i,j}}(\frac{\partial T}{\partial y}|_{\vec{x}_{i,j+1}} - \frac{\partial T}{\partial y}|_{\vec{x}_{i,j-1}}).$$

where S is the sign function, it is used to ensure that the jumps is computed from solid to liquid phase. Then we do the extension process using the advection equations as follows to avoid numerical difficulties in solving level set equation and they will still be denote as V_x and V_y.

$$(V_x)_t + S(\eta\eta_x)(V_x)_x = 0,$$

$$(V_y)_t + S(\eta\eta_y)(V_y)_y = 0.$$

5.2.5 Équation de Level set

We use a second order TVD scheme to update the function η at time step δt_h with superbee flux limiter g .

$$g(q) = \max(0, \min(1, 2q), (2, q)).$$

The discretization scheme reads,

$$\begin{cases} \eta_{i,j}^{t+\delta t_h} = \eta_{i,j}^t - \dfrac{\delta t_h}{h}(A_x^+ \eta_{i-1/2,j}^t + A_x^- \eta_{i+1/2,j}^t) \\ \qquad - \dfrac{\delta t_h}{h}(B_{i+1/2,j}^t - B_{i-1/2,j}^t) \\ \qquad - \dfrac{\delta t_h}{h}(A_y^+ \eta_{i,j-1/2}^t + A_y^- \eta_{i,j+1/2}^t) \\ \qquad - \dfrac{\delta t_h}{h}(B_{i,j+1/2}^t - B_{i,j-1/2}^t) \end{cases}$$

The detail expressions of the symbols used here see in the appendix.

Then we need to reinitialize η to be an exact distance function from the evolving front γ at each timestep. Like in [66], this is accomplished by iterating the equation

$$\eta_t = S_\epsilon(\eta_0)(1- \mid \vec{\bigtriangledown}\eta \mid),$$

to steady state, where $\eta(\vec{x}, 0) = \eta_0(\vec{x})$. S_ϵ is given by

$$S_\epsilon(\eta_0) = \frac{\eta_0}{\sqrt{\eta_0^2 + \epsilon^2}}.$$

From the distance function η, we determine the points on the boundary γ. They are obtained by piecewise linear interpolation. Then we approximate γ by connecting these points in the order such that the line is of minimum length.

5.2.6 Calcul de la température

Here we solve the elliptic equations,

$$(1 - \delta t_p \triangle)T^{n+1,\omega} = T^{n,\omega}, \quad \vec{x} \in \omega \tag{5.5}$$

$$(1 - \delta t_p \triangle)T^{n+1,\omega^c} = T^{n,\omega^c}, \quad \vec{x} \in \omega^c \tag{5.6}$$

where $T^{n,\omega}$ denotes the approximation of $T(t_n,\cdot)$ on ω while T^{n,ω^c} being the approximation of $T(t_n,\cdot)$ on ω^c, $t_n = n\delta t_p$, $n \in Z$.

Let $\mathbf{U}_j^\Omega$ be an approximation of periodic space $H_p^1(\Omega)$, $\{\phi_\alpha^\Omega\}_\alpha$ be an orthonormal basis. It is constructed by the periodic spline scaling function[60] with tensor product. One resolves the equations by the smooth fictitious domain method proposed in [40], which has a better convergence rate compared to classical fictitious domain method verified in the previous chapter. Note $T_j^{n+1,\omega} = \sum_\alpha \mathbf{T}_j^{n+1,\omega}\Phi_\alpha^\Omega$, thus we get the following linear system,

$$\begin{cases} \text{Find } \mathbf{T}_j^{n+1,\omega} \in R^{4^j}, \Lambda_{j'}^{n+1,\omega} \in R^{2^{j'}}, \text{ such that} \\ \mathbf{T}_j^{n+1,\omega} + C^{n+1,\omega}\Lambda_{j'}^{n+1,\omega} = F_j^n, \\ D^{n+1}\mathbf{T}_j^{n+1,\omega} = G_{j''}^{n+1}, \end{cases} \tag{5.7}$$

here

$$(F_j^n)_\alpha = (T^n, \theta_\alpha^\Omega)_{L^2(\Omega)},$$

$$C_{\alpha\alpha'}^{n+1,\omega} = (\zeta_{\gamma_{n+1}^{\text{ex}}}\theta_\alpha^\Omega, \phi_{\alpha'}^{\Gamma_{n+1}^{\text{ex}}})_{L^2(\Gamma_{n+1}^{\text{ex}})},$$

$$D_{\alpha''\alpha}^{n+1} = (B_{\gamma_{n+1}}\phi_\alpha^\Omega, \phi_{\alpha''}^{\gamma_{n+1}})_{L^2(\gamma_{n+1})},$$

$$(G_{j''}^{n+1})_{\alpha''} = (g^{n+1}, \phi_{\alpha''}^{\gamma_{n+1}})_{L^2(\gamma_{n+1})},$$

with $\theta_\alpha^\Omega = (1-\delta t_p\triangle)\Phi_\alpha^\Omega$, the boundary γ_{n+1} at time $(n+1)\delta t_p$, the exteterior control boundary Γ_{n+1}^{ex}, Dirichlet boundary condition g^{n+1} of temperature on γ_{n+1}. $\zeta_{\Gamma_{n+1}^{\text{ex}}}$ and $B_{\gamma_{n+1}}$ are the restriction operator from Ω to Γ_{n+1}^{ex} and γ_{n+1} respectively. When one intends to update the temperature T_j^{n+1,ω^c}, only the matrix C^{n+1,ω^c} needs to be redone by replacing Γ_{n+1}^{ex} with Γ_{n+1}^{in}, $\phi_{\alpha'}^{\Gamma_{n+1}^{\text{ex}}}$ by $\phi_{\alpha'''}^{\Gamma_{n+1}^{\text{in}}}$. The approximation level on Ω, Γ_{n+1}^{ex}, γ_{n+1}, Γ_{n+1}^{in} are j, j', j'', j''' respectively. The position of curve γ_n, Γ_n^{ex} and Γ_n^{in} is illustrated in plot 5.1. We use the Uzawa algorithm for (5.7), then it is equivalent to solve the following system

$$\begin{cases} \text{Find } \mathbf{T}_j^{n+1,\omega} \in R^{4^j}, \Lambda_{j'}^{n+1,\omega} \in R^{2^{j'}}, \text{ such that} \\ (D^{n+1}C^{n+1,\omega})^t D^{n+1}C^{n+1,\omega}\Lambda_{j'}^{n+1,\omega} = (D^{n+1}C^{n+1,\omega})^t D^{n+1}F_j^n - (D^{n+1}C^{n+1,\omega})^t G_{j''}^{n+1}, \\ \mathbf{T}_j^{n+1,\omega} = F_j^n - C^{n+1,\omega}\Lambda_{j'}^{n+1,\omega}. \end{cases}$$

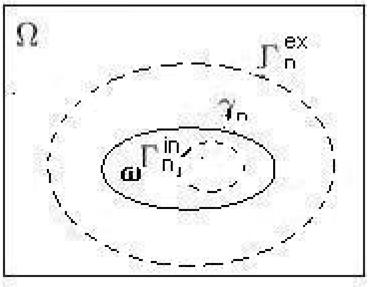


Figure 5.1: Geometry

The orthogonal basis $\phi^{\gamma_n}_{\alpha''}$ on γ_n is constructed by map $\upsilon_n : [0,1] \to R^2$ [13] such that

$$\gamma_n = \upsilon_n([0,1]).$$

Thus we get

$$\forall s \in \gamma_n, \quad \phi^{\gamma_n}_0(s) = \phi^{[0,1]}_0(\upsilon_n^{-1}(s)).$$

The term of matrix D^{n+1} reads,

$$D^{n+1}_{\alpha''\alpha} = \int_{[0,1]} \Phi^{\Omega}_{\alpha}(\upsilon_{n+1}(\tau))\phi^{[0,1]}_{\alpha''}(\tau) \mid J_{\upsilon_{n+1}}(\tau) \mid d\tau.$$

From [60], it is know that the coefficients with scaling function are related to their point value by interpolation filter. We calculate the point values of $\phi^{\Omega}_{\alpha}(\upsilon_{n+1}(\tau)) \mid J_{\upsilon_{n+1}}(\tau) \mid$ at $\{\tau = \frac{i}{2^{j''+r}}, i = 0, \cdots, 2^{j''+r} - 1, r \in \mathbb{Z}\}$. The reason to calculate at scale $j'' + r$ is to upgrade the precision of integration, usually $r = 2$. Then the decomposition process is applied to get the coefficient at scale j''. The calculation of matrix $C^{n+1,\omega}$, G^{n+1} does in the same way.

The control boundary Γ^{ex}_{n+1} is constructed by moving the points on γ_{n+1} in the direction of outer normal vector with distance δd^{ex} while the distance between γ_{n+1} and Γ^{in}_{n+1} is δd^{in}. For convenience, we usually take Γ^{ex}_{n+1} and Γ^{in}_{n+1} in the form of circle.

The following quadrature formula, which is an extension of 4th order in one dimension, is used to calculate the element in matrix F^n

$$\int_{\omega} f(x)dx = h^2 \sum_{i=1}^{N+1} \sum_{j=1}^{N+1} C(i)C(j) f_{i,j},$$

where $0 = x_1 < x_1 < \cdots < x_N < x_{N+1} = 1$, $h = \frac{1}{N}$, $x_i = (i-1)h$, $i = 1, \cdots, N+1$, $f_{i,j} = f(x_i, x_j)$. The coefficient $C(\cdot)$ is defined as following

$$C(i) = \begin{cases} 3/8, i = 1, N+1, \\ 7/6, i = 2, N, \\ 23/24, i = 3, N-1, \\ 1, (other). \end{cases}$$

5.2.7 Schéma des algorithmes

(1) Initialize the temperature T on Ω and distance function η.

(2) Calculate the velocity V on Ω and do extension using advection equations, determine the time step δt_h by dividing δt_p by m such that δt_h satisfies the CFL condition, $m \in I\!N$.

(3) Update η by resolving its governing equation and reinitialize it. This procedure repeats m time.

(4) Update the temperature with the new front.

(5) Repeat the process (1)-(4).

5.3 Résultats numériques

The first two examples, where exact solutions exist, are used to illustrate the efficiency of our approach. They are one phase problems and we only consider the temperature evolution in domain ω. The next examples, a true two phase problem, deal with more physical situation.

5.3.1 Interface plane mobile

Here $\omega = [x_0, X_0] \times [0,1]$ and $\Omega = [0,1] \times [0,1]$. $\gamma_f = \{(x,y); x = X_0, y \in [0,1]\}$ is a fixed boundary where the boundary conditions are $T = e^{t-X_0} - 1$ while $\gamma_m = \{(x,y); x = x_0, y \in [0,1]\}$ is a moving boundary where the boundary conditions is

$T = 0$. The initial temperature is $T_0(x, y) = e^{t_0 - x} - 1$ on $\omega_1 = [t_0, 1] \times [0, 1]$ while $T = 0$ on the domain $\Omega \setminus \omega_1$. The exact solution and moving interface are given by

$$\begin{cases} T(t, x, y) = e^{t-x} - 1, \\ \gamma_m(t) = \{(x, y); x = t, y \in [0, 1]\}. \end{cases}$$

We take $t_0 = 0.3$, $X_0 = 0.75$, $j = 6$, spline order $m = 4$, $\delta_p = \delta t_h = 0.0004$. The smooth fictitious domain method is used with a control boundary $\Gamma_n = \{(x, y); x = x_0 - 0.1, y \in [0, 1]\}$. The initial geometry and numerical results are on figure 5.2 and 5.3. We see that with the smooth fictitious domain method, the temperature is better resolved in the vicinity of the moving boundary resulting in a higher precision in the evaluation of the velocity of the moving boundary.

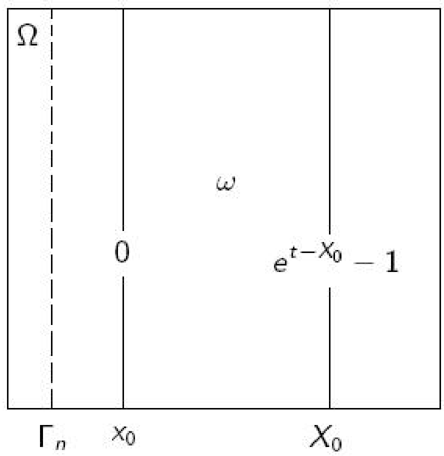
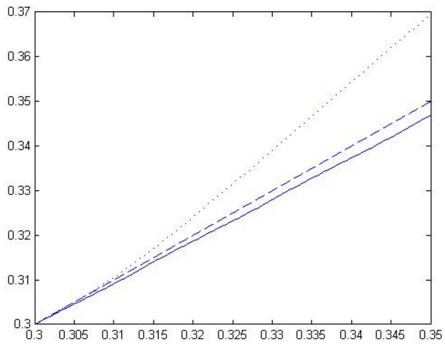


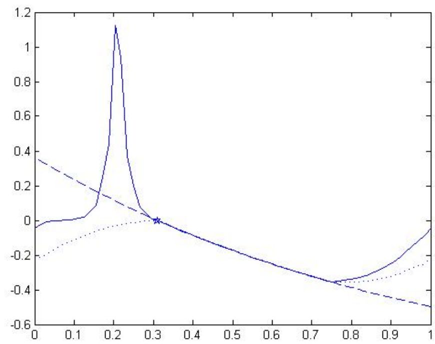
Figure 5.2: geometry of the planar problem

5.3.2 Interface circulaire

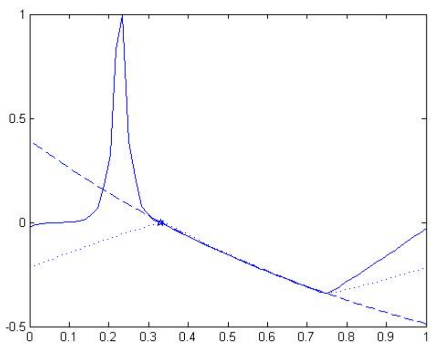
Here ω is a ring shaped domain centered around the point $(0.5, 0.5)$ and $\Omega = [0, 1] \times [0, 1]$. The fixed interior boundary γ_f is a circle of radius R_f while the moving



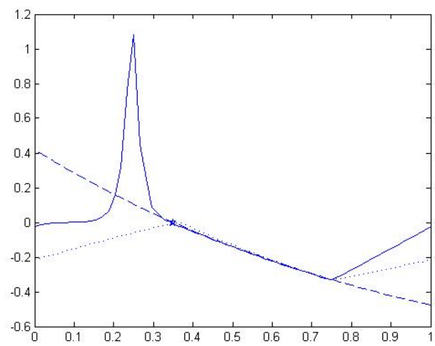
(a) the position of x_0 at different time, dashed line: exact, dot line: classical fictitious domain method, solid line: smooth fictitious domain method



(b) t=0.31



(c) t=0.3316



(d) t=0.35

Figure 5.3: (b),(c),(d): continuous line cross-section of computed solution with smooth fictitious domain method; dashed line exact solution; dot line cross-section of computed solution with classical fictitious domain method; pentagram for the position of the moving boundary

exterior boundary γ_m is a circle of radius R_m. The problem reads

$$\begin{cases} \dfrac{\partial T}{\partial t} - \triangle T = (2 - \dfrac{1}{\rho}) e^{R_{\max} - \rho - t}, \\ T = 1 - e^{R_{\max} - R_f - t} \quad \text{on} \quad \gamma_f, \\ T = 0 \quad \text{on} \quad \gamma_m, \\ T(t_0, \rho) = 1 - e^{R_{\max} - \rho - t_0}, \\ R_m(t_0) = R_{\max}. \end{cases}$$

where $\rho = \sqrt{x^2 + y^2}$. Its solution is given by

$$\begin{cases} T(t, \rho) = 1 - e^{R_{\max} - \rho - t}, \\ R_m(t) = R_{\max} - t. \end{cases}$$

The parameters are $R_{\max} = 0.4$, $R_f = 0.15$, $t_0 = 0$. The simulation is performed using our methods with splines of order $m = 4$ and scale parameters $j = 4, 5, 6, 7$, $j' = j'' = 4$, $\delta_p = 0.001$, δ_h automatically chosen to satisfy the CFL condition. For the smooth fictitious domain method, the control boundary Γ_n is a circle of radius 0.45. We see the initial geometry on figure 5.4. The least square errors between the exact boundary and the numerical boundary versus scale j at time $t = 0.1$ for the classical and smooth fictitious domain method are showed in figure 5.5.

In [54],[65], the authors focus on the behavior of a sinusoidal perturbation on the initial interface. It is shown that it is amplified with the time evolution. When perturbing the exterior boundary, one has

$$R_m(t_0) = \bar{R}_m + \varepsilon_p sin(K\theta).$$

with $\bar{R}_m = 0.4$, $\varepsilon_p = 2.2e - 2$ and $K = 10$. The results are given on figure 5.6.

5.3.3 Simulation de problémes physiques

We consider the problem

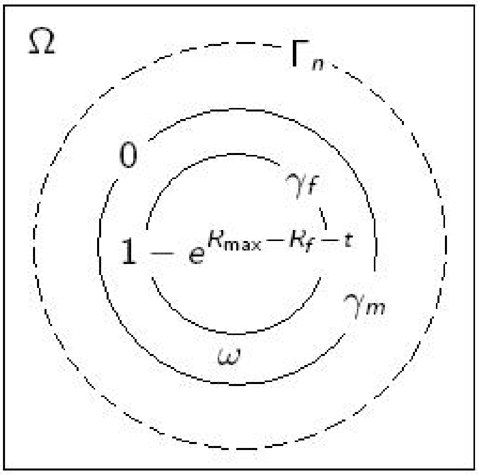


Figure 5.4: geometry of circle problem

$$\begin{cases} \frac{\partial T}{\partial t} = \triangle T, \vec{x} \in \omega \text{ or } \omega^c \\ T(t,\vec{x}) = -\epsilon_c k - \epsilon_v V \quad \text{on interfacial surface} \\ T(0,\vec{x}) = 0, \vec{x} \in \omega \\ T(0,\vec{x}) = -0.5, \vec{x} \in \omega^c \end{cases}$$

These equations simulate supercooling, where a small frozen seed of material($T = 0$ in ω) is placed in a surrounding region of under cooling liquid($T = -0.5$ in ω^c). The boundary temperature is determined by the Gibbs-Thomson relation, which is a combination of curvature and velocity on interfacial surface.

The initial interfacial shape is given by,

$$\begin{cases} x(s) = 0.5 + (R + Pcos(4\alpha))cos(\alpha), \\ y(s) = 0.5 + (R + Pcos(4\alpha))sin(\alpha), \end{cases}$$

where $R = 0.1$ and $P = 0.02$, $\alpha \in [0, 2\pi]$. The parameters used to get the boundary condition are, $\epsilon_C = 0.002$, $\epsilon_V = 0.002$. The two control boundaries, interior Γ_n^{in} and exterior Γ_n^{ex}, are circles of radius dist$\{(0.5, 0.5), \gamma_n\}/2$, dist$\{(0.5, 0.5), \gamma_n\} + 0.1$ respectively, where γ_n is the real boundary. The geometry is on figure . The other parameters are $m = 4$, $\delta_p = 0.005$, δ_h chosen by CFL condition, $j' = j'' = j''' = 4$,

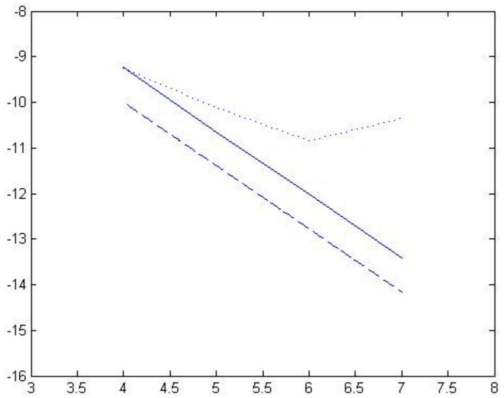


Figure 5.5: Circular interface case. Least square error versus scale j in semi-logarithms scale, solid line: smooth method, dotted line: classical method, dashed line: line of slope -2log(2)

$j = 4, 5, 6, 7$. We use this example to observe the evolution of the interface over time with different grid resolutions, see in figure 5.8. The surface tension coefficient ϵ_c has a stabilizing effect on the unstable problem noted in [14]. To observe this phenomenon, we set $\epsilon_V = 0$, $\epsilon_C = 0, 0.0005, 0.001$ in plot 5.9. The last example in figure 5.10 concerns the effect of diffusion coefficient $\kappa = 1, 2, 3, 4$ on the evolution of interface.

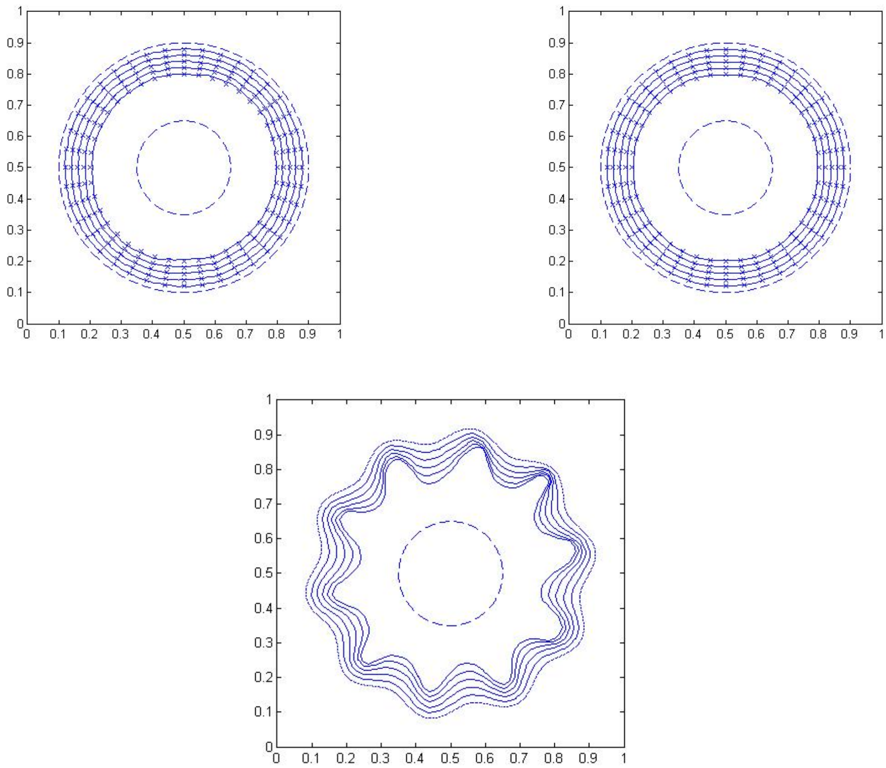


Figure 5.6: Evolution of the interface for $j = 6$. Top: left using classical fictitious domain method, right using smooth one. Bottom: under the situation of perturbed interface with smooth method. Lines stand for computational interface, dashed line for initial boundary, XX for exact interface. Time levels shown are in increment of 0.02 up to a final time of 0.1.

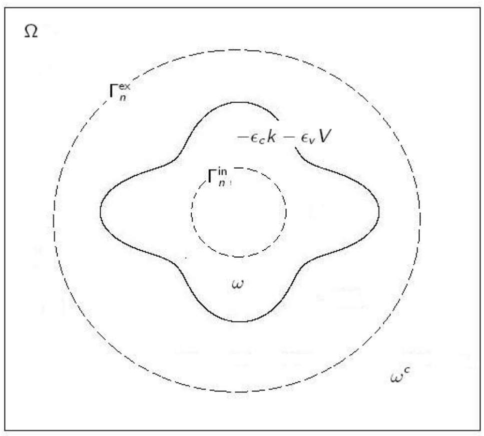


Figure 5.7: geometry of a two phase problem

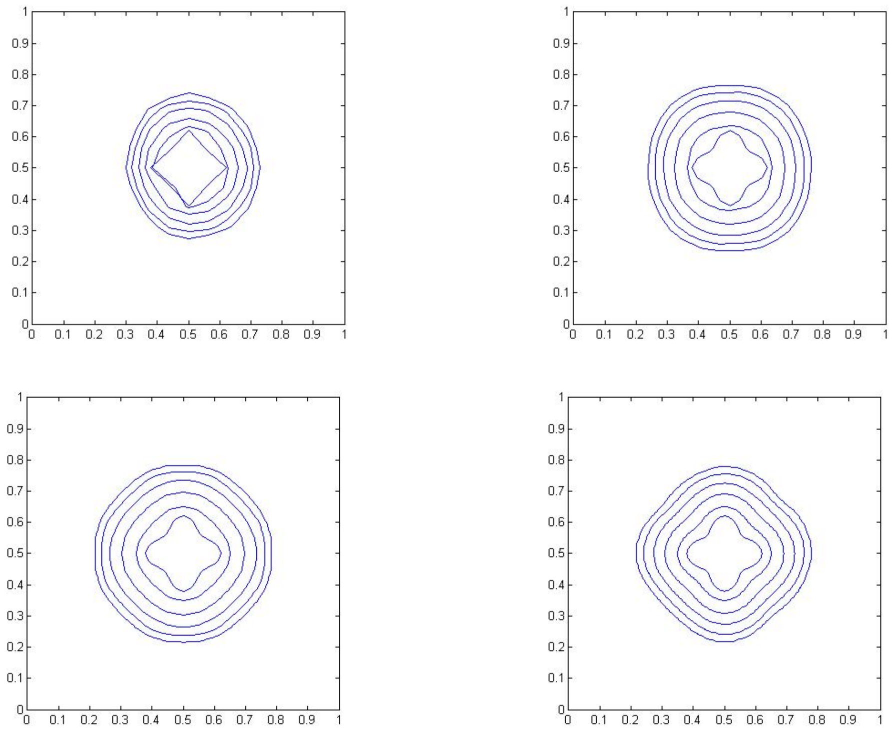


Figure 5.8: Interface evolution with j=4,5,6,7 from the top left to the bottom right. Time levels shown are in increment of 0.02 up to a final time of 0.1.

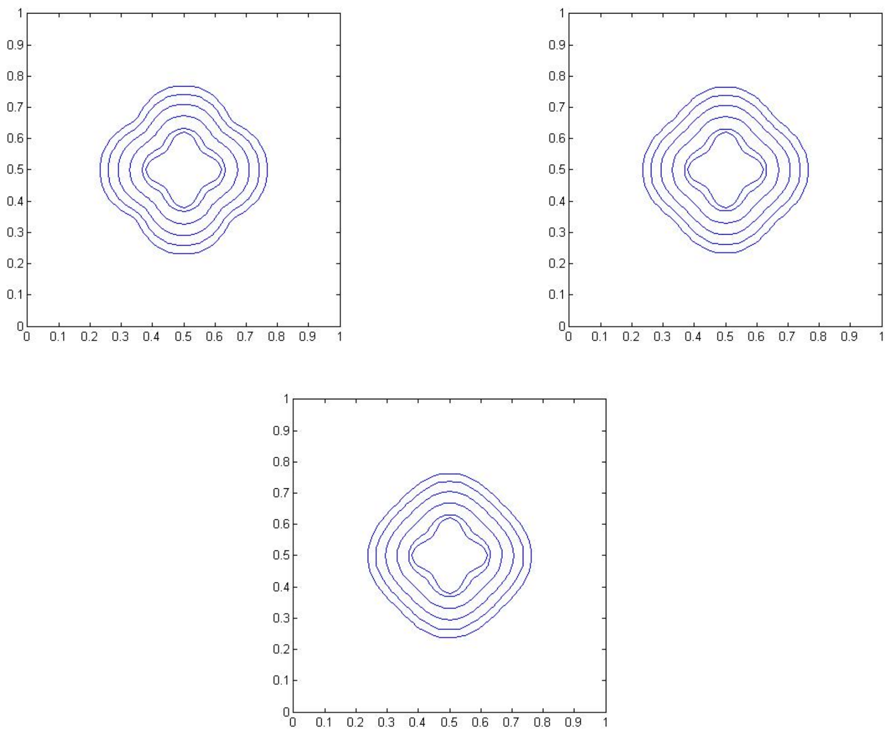


Figure 5.9: Effect of surface tension coefficient ϵ_c on moving boundary. For all plots, $j = 6$, $j' = j'' = j''' = 4$, $\epsilon_V = 0$. $\epsilon_C = 0, 0.0005, 0.001$, time increment 0.02, the final time 0.1

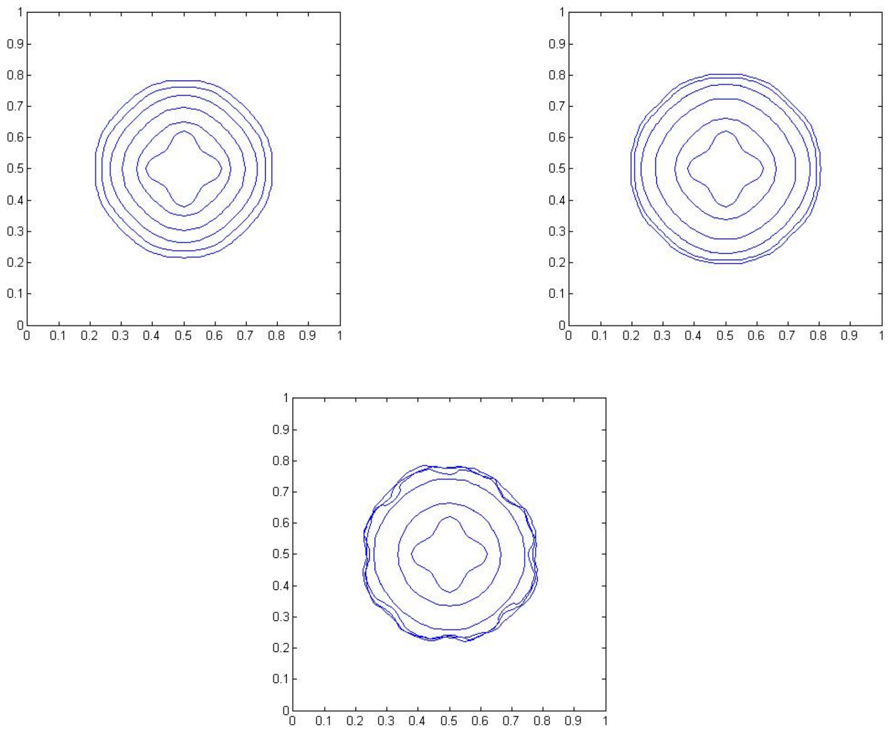


Figure 5.10: Effect of different diffusion coefficient κ. $\kappa = 1, 2, 4$ from top left, top right to bottom. For all plots, $j = 6$, $j' = j'' = j''' = 4$, $\epsilon_V = 0.002$, $\epsilon_C = 0.002$, time increment 0.02, the final time 0.1

Conclusion et perspectives

Notre contribution à la simulation de problèmes de Stefan peut être résumée comme suit:

(1) Une nouvelle méthode numérique pour l'approximation de la solution de problèmes elliptiques sur un domaine général a été développée, analysée et implémentée. Elle est basée sur divers ingrédients:

– Le prolongement lisse qui permet d'étendre une fonction connue sur un domaine général ω à un plus grand domaine Ω en utilisant les constructions d'analyse multirésolution sur l'intervalle.

–La méthode de domaine fictif lisse qui utilise une frontière de contrôle différente de celle du domaine initial.

– L'approximation de Petrov-Galerkin à base d'ondelettes et les estimations d'erreurs intérieurs et globales associeés. Les préconditionneurs en ondelettes sont également utilisés et les estimations théoriques associées sont développées.

(2) Une méthode numérique complète pour l'approximation de problème de Stefan à partir de:

- L'algorithme ci-dessus couplé à une discrétisation en temps de type différences finies.

- La formulation level set pour l'évolution des frontières et son approximation à l'aide des différences finies.

(3)Des simulations tests divers et la validation numérique de résultats théoriques, incluant la simulation de problème de Stefan à 2 phases.

Diverses orientations de travail restent à explorer:

(1) Les simulations adaptive sont souhaitables et pourraient être très efficaces pour des problèmes de Stefan. Elles sont théoriquement possibles mais exigent comme d'habitude de résoudre les difficultés techniques de programmation.

(2) Les équations de level set peuvent bénéficier de formulations faibles, surtout quand la normale, la courbure ou d'autres grandeurs géométriques sont utilisées.

(3) Certaines de nos estimations d'erreur intérieure n'ont été obtenues que dans des situations en dimension 1. Leur dérivation en dimension 2 est encore ouverte.

(4) Notre procédure de prolongement est jusqu'à présent limitée aux domaines convexes, contrairement à d'autres méthodes comme la méthode spectrale[30]. La généralisation de notre procedure et sa comparaison avec les autres méthodes existantes, en particulier en 2D, restent à faire.

Chapter 6

Annexe

6.1 Analyse multirésolution spline(section 1.1.3)

The polynomial P_m of order m is given by the formula:

$$\frac{P_{m-1}(sin^2 z)}{(sinz)^{2m}} = \sum_{l \in z} \frac{1}{(z + l\pi)^{2m}}$$

It is calculated by the recurrence relation as follows:

If $P_m(x) = \sum_{i=0}^{m} a_i^m x^i$, then

$$\begin{aligned}
&a_0^0 = 1, \quad a_1^1 = -\frac{2}{3}, \quad a_0^m = 1 \\
&a_m^m = -\frac{2}{m(2m+1)} a_{m-1}^{m-1} \\
&a_i^m = \frac{1}{m(2m+1)}[(m-i)(2m-2i+1)a_i^{m-1} - 2(m-i+1)^2 a_{i-1}^{m-1}], \quad 1 \le i \le m.
\end{aligned}$$

6.2 Coefficients d'échelle $H^0_{k,l}$, $H^{0,\sharp}_{k,l}$ pour $N = 5$(section 4.1.1)

The coefficients $H^0_{k,l}$, $H^{0,\sharp}_{k,l}$ of edge scaling functions, defined in section 1.2.2, are given in table 6.1, 6.2.

Table 6.1: The coefficients $H^0_{k,l}$ and $H^{0,\sharp}_{k,l}$ with $N = 5$

	l	$H^0_{k,l}$	$H^{0,\sharp}_{k,l}$
k=0	0	6.629995e-001	9.302491e-001
	1	5.590853e-001	3.488878e-001
	2	-3.638896e-001	1.098578e-001
	3	-3.190292e-001	2.701026e-002
	4	-8.577516e-002	7.897330e-003
	5	7.938923e-002	7.300852e-003
k=1	0	-1.615977e-001	-3.099501e-001
	1	7.429922e-001	5.987214e-001
	2	5.771522e-001	6.208411e-001
	3	2.773360e-001	3.720165e-001
	4	1.530007e-002	1.451060e-001
	5	-1.063943e-001	6.263262e-003
	6	-1.216770e-002	1.687072e-002
	7	1.126647e-002	1.562116e-002
k=2	0	2.576682e-001	1.138401e-001
	1	-3.995181e-002	-3.641832e-001
	2	1.290659e-002	1.228916e-002
	3	4.163709e-001	4.117590e-001
	4	6.469375e-001	5.954973e-001
	5	5.608421e-001	5.457944e-001
	6	4.995701e-003	1.756416e-001
	7	-1.569483e-001	-1.751671e-002
	8	-2.009486e-002	2.376581e-002
	9	1.860649e-002	2.200555e-002

6.3 Moments de $\phi^0_{0,k}$ or $\phi^{0,\sharp}_{0,k}$ pour $N = 5$(section 4.1.5)

The moments $\int x^l \phi^0_{0,k}$, $l = 0, \cdots, N-1$(resp $\int x^l \phi^{0,\sharp}_{0,k}$) of the functions

$$\phi^0_{0,k}$$

(resp $\phi^{0,\sharp}_{0,k}$) for $N = 5$ are given in table 6.3, 6.4, 6.5 and 6.6.

6.4 Discrétisation de l'équation de level set(section 5.2.5)

The expressions of A_x^+, A_x^-, A_y^+, and A_y^- introduced in section 5.2.5 are given by:

$$A_x^+ \eta^t_{i-1/2,j} = \begin{cases} V^t_{x,i-1/2,j}(\eta^t_{i,j} - \eta^t_{i-1,j}), \text{ if } V^t_{x,i-1/2,j} > 0, \\ 0, \text{ otherwise }, \end{cases}$$

$$A_x^- \eta^t_{i+1/2,j} = \begin{cases} V^t_{x,i+1/2,j}(\eta^t_{i+1,j} - \eta^t_{i-1,j}), \text{ if } V^t_{x,i+1/2,j} < 0, \\ 0, \text{ otherwise }, \end{cases}$$

$$A_y^+ \eta^t_{i,j-1/2} = \begin{cases} V^t_{y,i,j-1/2}(\eta^t_{i,j} - \eta^t_{i,j-1}), \text{ if } V^t_{y,i,j-1/2} > 0, \\ 0, \text{ otherwise }, \end{cases}$$

$$A_y^- \eta^t_{i,j+1/2} = \begin{cases} V^t_{y,i,j+1/2}(\eta^t_{i,j+1} - \eta^t_{i,j}), \text{ if } V^t_{y,i,j+1/2} < 0, \\ 0, \text{ otherwise }. \end{cases}$$

Moreover, $B^t_{i-1/2,j}$ and $B^t_{i,j-1/2}$ are

$$B^t_{i-1/2,j} = \frac{1}{2}|V^t_{x,i-1/2,j}|(1 - \frac{\delta t_h}{h}|V^t_{x,i-1/2,j}|)W^t_{x,i-1/2,j},$$
$$B^t_{i,j-1/2} = \frac{1}{2}|V^t_{y,i,j-1/2}|(1 - \frac{\delta t_h}{h}|V^t_{y,i,j-1/2}|)W^t_{y,i,j-1/2},$$

where

$$W^t_{x,i-1/2,j} = g(q^t_{x,i-1/2,j})(\eta^t_{i,j} - \eta^t_{i-1,j}),$$
$$W^t_{y,i,j-1/2} = g(q^t_{y,i,j-1/2})(\eta^t_{i,j} - \eta^t_{i,j-1}),$$

with

$$q^t_{x,i-1/2,j} = \begin{cases} \dfrac{\eta^t_{i-1,j} - \eta^t_{i-2,j}}{\eta^t_{i,j} - \eta^t_{i-1,j}}, \text{ if } V^t_{x,i-1/2,j} > 0, \\ \dfrac{\eta^t_{i+1,j} - \eta^t_{i,j}}{\eta^t_{i,j} - \eta^t_{i-1,j}}, \text{ if } V^t_{x,i-1/2,j} < 0, \end{cases}$$

$$q^t_{y,i,j-1/2} = \begin{cases} \dfrac{\eta^t_{i,j-1} - \eta^t_{i,j-2}}{\eta^t_{i,j} - \eta^t_{i,j-1}}, \text{ if } V^t_{y,i,j-1/2} > 0, \\ \dfrac{\eta^t_{i,j+1} - \eta^t_{i,j}}{\eta^t_{i,j} - \eta^t_{i,j-1}}, \text{ if } V^t_{y,i,j-1/2} < 0. \end{cases}$$

Table 6.2: Continuation of table of table 6.1

	l	$H^0_{k,l}$	$H^{0,\sharp}_{k,l}$
k=3	0	-8.955159e-002	-4.098817e-002
	1	3.895617e-002	1.431056e-001
	2	4.751157e-003	-6.777712e-002
	3	-6.596189e-002	-1.740980e-001
	4	1.520881e-002	-9.559133e-002
	5	2.466645e-001	1.512981e-001
	6	7.199998e-001	6.357422e-001
	7	6.131240e-001	6.823643e-001
	8	1.537319e-002	1.963091e-001
	9	-1.721948e-001	-3.301997e-002
	10	-2.083859e-002	2.833562e-002
	11	1.929514e-002	2.623688e-002
k=4	0	1.546583e-002	7.965030e-003
	1	-6.559247e-003	-2.796122e-002
	2	2.728094e-003	1.706239e-002
	3	1.830646e-002	2.928655e-002
	4	1.708296e-002	2.857807e-003
	5	2.156026e-002	-4.686793e-002
	6	-3.692524e-002	-1.702677e-001
	7	2.045335e-001	3.028148e-002
	8	7.233636e-001	6.351733e-001
	9	6.327590e-001	7.207789e-001
	10	1.649999e-002	1.995470e-001
	11	-1.751674e-001	-3.864256e-002
	12	-2.109312e-002	2.947288e-002
	13	1.953081e-002	2.728991e-002

Table 6.3: First N moments of $\phi^0_{0,k}$

x^l	$\phi^0_{0,0}$	$\phi^0_{0,1}$	$\phi^0_{0,2}$
l=0	-6.098748e-002	1.160713e+000	1.007806e+000
l=1	-5.286166e-001	7.253578e-001	1.865242e+000
l=2	-6.137187e-001	4.486587e-001	3.436344e+000
l=3	-5.056761e-001	-3.037541e-002	5.781685e+000
l=4	9.464128e-002	-1.073163e+000	7.750883e+000

Table 6.4: Continuation of table 6.3

x^l	$\phi^0_{0,3}$	$\phi^0_{0,4}$
l=0	1.008060e+000	1.000413e+000
l=1	2.902325e+000	3.885262e+000
l=2	8.351550e+000	1.508903e+001
l=3	2.347376e+001	5.806443e+001
l=4	6.275833e+001	2.192554e+002

Table 6.5: First N moments of $\phi^{0,\sharp}_{0,k}$

x^l	$\phi^{0,\sharp}_{0,0}$	$\phi^{0,\sharp}_{0,1}$	$\phi^{0,\sharp}_{0,2}$
l=0	1.113588e+000	1.084835e+000	1.069977e+000
l=1	6.513167e-001	1.964814e+000	3.205332e+000
l=2	6.851144e-001	3.566928e+000	9.543633e+000
l=3	1.003691e+000	7.418776e+000	2.892215e+001
l=4	1.846721e+000	1.763328e+001	9.173689e+001

Table 6.6: Continuation of table 6.5

x^l	$\phi_{0,3}^{0,\sharp}$	$\phi_{0,4}^{0,\sharp}$
l=0	1.025369e+000	1.001581e+000
l=1	4.204360e+000	5.124434e+000
l=2	1.722262e+001	2.621836e+001
l=3	7.101418e+001	1.346789e+002
l=4	2.968720e+002	6.973029e+002

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